BLACKIE'S

ALGEBRA

FOR BEGINNERS

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PREFACE.

This book aims at giving a thorough grounding in the elementa rules of Algebra, by means of simple explanations enforced by abunda illustrative examples. It follows the syllabus of the first stage Mathematics of the Board of Education applicable to schools an classes other than elementary, containing, however, in addition, section on Square Root and a section on More Difficult Factors. Thes two sections are added so that the book may meet exactly the requir ments in Algebra of the course laid down in the New Extra Subjec programme of the Irish Board of Education.

Great care has been exercised in the selection and arrangement the exercises. At the end of the book will be found a number of te papers, from which teachers may set additional examples on the boo work, and which should prove useful in periodical revisions.

Complete answers are appended.

ELEMENTARY ALGEBRA.

DEFINITIONS.

Algebra is the science which treats of numbers by means of letters which represent the numbers, and certain signs which denote the operations to be performed on the numbers and the manner in which they are connected with one another.

The principal signs used in algebra are as follows:—

$+$ the sign of **addition**, read **plus**.

$-$ the sign of **subtraction**, read **minus**.

$\times$ the sign of **multiplication**, read **into** or **multiplied by**.

$\div$ the sign of **division**, read **by** or **divided by**.

$=$ the sign of equality, read **equals** or **is equal to**.

$\sim$ the **difference** sign, and denotes that the smaller of two quantities between which it is placed is to be subtracted from the greater.

$\therefore$ denotes **therefore**.

$\because$ denotes **because**.

$>$ denotes **is greater than**; thus, $3 > 2$.

$<$ denotes **is less than**; thus $2 < 3$.

Note.—The opening between the lines in the two preceding signs is turned towards the greater of the two numbers between which they may be placed.

$\sqrt{\ }$ is the **radical sign**.

$\sqrt[2]{\ }$ or simply $\sqrt{\ }$ is the sign of the **square root**, or second root (generally read the **root**).

$\sqrt[3]{\ }$ is the sign of the **cube root**, or **third root**.

$\sqrt[4]{\ }$ is the sign of the **fourth root**.

$\sqrt[n]{\ }$ is the sign of the **nth root**, where **n** may be any number.

( ), { }, [ ], are called **brackets**, and are used to denote that all the quantities within them are to be treated as one quantity.

The sign of multiplication given above is not much used in Algebra. When two letters are placed close together without any sign between them, this means that the numbers which these letters represent are to be multiplied together. Thus $a \times b$ is generally written ab, so also $8 \times b$ is generally written $8b$. If a stands for 5 and b stands for 8, then ab stands for 5×8, that is 40. Sometimes a point is used instead of the sign $\times$. Thus $a \times b$ and $a.b$ mean the same thing. It is evident that when numbers which are represented by figures are to be multiplied together the sign $\times$ should be used, as 58 does not mean 5 *multiplied by* 8 but *fifty-eight*. Nor should the point be used as a sign of multiplication where there is any chance of it being mistaken for the *decimal point*.

As in Arithmetic, so in Algebra, when one number is placed over another with a line between them, this signifies that the upper number is to be divided by the lower. Thus $\frac{a}{b}$ means *a divided by b*. If a stands for 12 and b stands for 3, then $\frac{a}{b}$ stands for $\frac{12}{3}$, that is 4.

The letters and signs used in Algebra are called **algebraical symbols**, and a collection of these is called an **algebraical expression**, or simply an **expression**. Those parts of an expression which are connected by the signs of addition or subtraction are called its **terms**. A **simple** expression has only one term. A **compound** expression has more than one term. A **binomial** expression has **two** terms, a **trinomial** has **three** terms, and so on. For example, $7a + 3ab - 5bc$ is a trinomial expression, of which the terms are $7a$, $3ab$, and $5bc$. It will be noticed that each of these terms is a simple expression.

The **factors** of an expression are those parts which are connected by the sign of multiplication. Thus, in the expression $ab \times (b + c)$ the factors are a, b, and $(b + c)$. When one factor of an expression is a number expressed by a figure it is called a **numerical coefficient**, or simply a **coefficient**, of the other factors. Thus, in the expression $5ab$, 5 is the coefficient of ab. A letter may also be used as a coefficient in an expression. Thus, in the expression ax the letter a may be called the coefficient of x.

A product which is formed by multiplying any quantity by itself any number of times is called a **power** of that quantity. Thus, 27 is the third power of 3, because it is got by multiplying three 3's together. A power of a quantity is expressed by writing the quantity with a number over it and to the right, to indicate the number of times that the quantity occurs in the product. Thus, a^4 means $a \times a \times a \times a$. The number over the quantity, as above, is called the **index** or **exponent** of the power. a, or as it is sometimes written, a^1, is the **first** power of a; a^2 is the **second** power or **square** of a; a^3 is the **third** power or **cube** of a; a^n is the n^{th} power of a. a^2 is read *a* **squared**, a^3 is read *a* **cubed**, a^n is read *a* **to the n^{th} power** or *a* **to the n^{th}**.

The **square root** of a given quantity is that quantity whose square or second power is equal to the given quantity. The **cube root** of a given quantity is that quantity whose cube or third power is equal to the given quantity. So also the **fourth root, fifth root,** &c., of a given quantity is that quantity whose fourth power, fifth power, &c., is equal to the given quantity.

The **number of dimensions** of a term is the sum of the indices of the letters composing it. Thus xy^2z^3 or $x^1y^2z^3$ is of six dimensions, or of the sixth degree. Numerical coefficients are not taken into account in reckoning the dimensions of a term. A **homogeneous** expression is one having all its terms of the same dimensions. Thus the expression $x^2 + 2xy + y^2$ is homogeneous, each term being of two dimensions.

Quantities which have the same letters involved in the same way are called **like** quantities.

Thus x^2y, $4x^2y$, $-3x^2y$, are **like** quantities.

a^3b^2c, $2a^2b^3c$, $-ab^2c^3$, are **unlike** quantities.

Quantities which have the same signs are said to have **like** signs.

The order of the terms of an expression may be changed without altering the value.

Thus, $a^2-bc+\sqrt{d}$, $\sqrt{d}+a^2-bc$, $-bc+a^2+\sqrt{d}$, $\sqrt{d}-bc+a^2$ have the same value. Each denotes that the sum of the square of a and the square root of d is diminished by the product of b and c.

Hence, to find the numerical value of an expression consisting of a number of positive and negative terms:—

Find the value of each term; collect all the positive terms into one sum, and the negative terms into another; then take the difference of these.

If the sum of the negative terms is more than that of the positive put $-$ before the result.

The reason of the last statement will be understood from the following: $a-b$ means that the quantity represented by b is to be subtracted from the quantity represented by a.

If $a=12$, and $b=8$, $a-b=12-8=4$.
But if $a=8$, and $b=12$, then $a-b=8-12=-4$.

Here we are subtracting a greater quantity from a less. Since $12=8+4$, we may subtract this 8 from 8 and the remainder is 0. But 4 still remains to be subtracted. This is represented thus, -4.

-4 is said to be a **negative quantity**.

Quantities with a $+$ prefixed (or without a sign prefixed) are termed **positive** quantities, *e.g.* 2, $+3$, a, $+b$.

Quantities with a $-$ prefixed are termed **negative** quantities, *e.g.* -2, $-a$, $-b$.

$12-8=4$ also means that 4 is the **algebraic sum** of the positive quantity 12, and the negative quantity -8.

$8-12=-4$ also means that -4 is the **algebraic sum** of the positive quantity 8, and the negative quantity -12.

We shall now work out a few examples as illustrations of the definitions and explanations which have been given.

If $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $f=0$, find the values of the following expressions:—

1. $a-b+c-d+e$.
2. $2a-3b+6c-8d+7e-5f$.
3. $\sqrt{e^2+bcd}-ae\sqrt{c^2+d^2}$.
4. $\dfrac{4a+6cd}{c+d^2}-\dfrac{6d+e^2}{c+d}$.

1. $a-b+c-d+e$
$=1-2+3-4+5$
$=9-6$
$=3$. Ans.

2. $2a-3b+6c-8d+7e-5f$
$=2.1-3.2+6.3-8.4+7.5-5.0$
$=2-6+18-32+35-0$
$=55-38$
$=17$. Ans.

3. $\sqrt{e^2+bcd} - ae\sqrt{c^2+d^2}$

$=\sqrt{5^2+2.3.4}-1.5\sqrt{3^2+4^2}$

$=\sqrt{25+24}-5\sqrt{9+16}$

$=\sqrt{49}-5\sqrt{25}$

$=7-5.5$

$=7-25$

$=-18.$ Ans.

4. $\dfrac{4a+6cd}{c+d^2}-\dfrac{6d+e^2}{c+d}$

$=\dfrac{4.1+6.3.4}{3+4^2}-\dfrac{6.4+5^2}{3+4}$

$=\dfrac{4+72}{3+16}-\dfrac{24+25}{3+4}$

$=\dfrac{76}{19}-\dfrac{49}{7}$

$=4-7$

$=-3.$ Ans.

Exercises I.

Find the numerical values of the expressions from 1 to 6 inclusive when $a=2$, $b=3$, $c=1$, $d=5$, $e=6$, $f=4$.

1. $3a+5b-2c.$
2. $5a+6b-7c-3d.$
3. $a+2b-3c+5d-e+f.$
4. $7ab-6bc+5cd-de.$
5. $4a-3ab+6bcd-2e-f.$
6. $2abc-ae+bf-ef+6ac.$

Find the numerical values of the expressions from 7 to 26 inclusive when $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, and $f=0$.

7. $6ac-5abde+7bcd+f.$
8. $13ab-16bf+4de.$
9. $20abc-2cde+bcde-def.$
10. $4ab-7ef+3df-6def+6d.$
11. $\dfrac{2a}{b}+\dfrac{3b}{2}-\dfrac{16c}{d}+\dfrac{15d}{e}.$
12. $ab+\dfrac{10bc}{de}+\dfrac{18df}{ac}.$
13. $\dfrac{9a+3b}{c}+\dfrac{a+5c}{d}-\dfrac{2d+4e}{7b}.$
14. $\dfrac{a+b}{b-a}+\dfrac{c+d}{d-c}+\dfrac{e+f}{e-f}.$
15. $\dfrac{ac-bc+de}{3cd-bd-ae}.$
16. $\dfrac{5ac+ade+5bc+bde}{5bd+2de+5e}.$
17. $a^2+2ab+b^2.$
18. $2a^2-4b^2+c^2+d^2.$
19. $a^2b+b^2c+abc+ab^2.$
20. $5ab^2-6a^2c+4bc^2-d^3.$
21. $a^2b^2-2ab^2c+b^2c^2+2abcd+c^2d^2-2bc^2d.$
22. $a^5-b^4c+c^3d^2-d^2e^2f-3abcde+ab^2cd.$
23. $\dfrac{7a^2+5b^2}{3a^2}-\dfrac{7bc-2c^2}{de+b^2}.$
24. $\dfrac{2a^2+ad+bd+2ab+b^2}{2b^2+2ab+ac+bc+ad+bd}.$
25. $\dfrac{a^2-2ab+b^2}{b^2-2bc+c^2}-\dfrac{d^2-2de+e^2}{c^2-2cd+d^2}.$
26. $\dfrac{8a^3+7b^3}{16}-\dfrac{111}{d^3-c^3}+\dfrac{d^2f^2}{5a^2+7c^2}.$
27. Find the value, when $a=3$ and $x=2$, of
$$\frac{8x^6+48ax^5+60a^2x^4-80a^3x^3-90a^4x^2+108a^5x-27a^6}{x^6-3ax^5+6a^2x^4-7a^3x^3+6a^4x^2-3a^5x+a^6}.$$

Find the numerical values of the expressions from 28 to 37 inclusive when $a=4$, $b=3$, $c=2$, and $d=0$.

28. $(a+b)^2-(b+c)^2+(c+d)^2.$
29. $a^2b^2-2ab(a+b)+(a+b)^2.$
30. $\dfrac{a(a+c)+b(a+d)}{b(a+c)+d(b+c)}.$
31. $\sqrt{(2ab+ac+bc-c)}.$

32. $\sqrt{(4a^2b^2+9b^2c^2-12ab^2c)}$.
33. $\sqrt{(b^2+9a-6b\sqrt{a})}$.
34. $\sqrt{(a+b+c)(a+b-c)(b+c-d)}$.
35. $\{(3b+4c)-2a\}^2$
36. $3a-\{2ab-(2b-c)^2\}$.
37. $2a\{3c(a+b)^2-4b\}$.
38. If $a=4$; $b=3$; $c=11$; $x=6$; find the numerical value of $\dfrac{\sqrt{ab+4x}-\sqrt{a^2+b^2}}{\sqrt{a^2b^2-4c}-\sqrt{7x+2(ab-1)}}$.

ADDITION.

CASE I.—To add like quantities with like signs.

RULE.—*Add all the coefficients of the several quantities together, annex the common letter or letters, and prefix, if necessary, the common sign.*

EXAMPLE 1. $5a+6a+3a+8a=+22a$ or simply $22a$.

EXAMPLE 2. $3ax+2ax+12ax=17ax$.

EXAMPLE 3. $-ab-2ab-5ab-6ab=-14ab$.

EXAMPLE 4. $-3xy^2-2xy^2-8xy^2-4xy^2=-17xy^2$.

Instead of arranging the quantities in a horizontal line, as in the above examples, we may arrange them vertically as in the following examples:—

EXAMPLE 5.	EXAMPLE 6.	EXAMPLE 7.
$13abc$	$-5ax^2y^2$	$-6\sqrt{ax}$
$14abc$	$-7ax^2y^2$	$-2\sqrt{ax}$
$2abc$	$-6ax^2y^2$	$-\sqrt{ax}$
$29abc$	$-18ax^2y^2$	$-9\sqrt{ax}$

CASE II.—To add like quantities with unlike signs.

RULE.—*Add all the coefficients of the positive quantities into one sum, and all the coefficients of the negative quantities into another; take the difference of these two sums, prefix, if necessary, the sign of the greater, and annex the common letters.*

EXAMPLE 1. $6a+4a-7a+a-5a+3a=2a$.

EXAMPLE 2. $-4abx^2y+7abx^2y-5abx^2y+abx^2y=-abx^2y$.

EXAMPLE 3. $5xy-xy+2xy-4xy-2xy=0$.

CASE III.—To add quantities which are not all like quantities.

RULE.—*Add together the like quantities by the preceding rules, and write down the other quantities with their proper signs.*

EXAMPLE 1. $4x-5y+3y-2x+4y=2x+2y$.

EXAMPLE 2. Add together the three expressions $13a-5b+c-13d$, $-2a+b+4c+2d$, and $-a-b+c-d$.

Collecting the like quantities together and arranging in a line we get:—

$13a-2a-a-5b+b-b+c+4c+c-13d+2d-d=10a-5b+6c-12d$.

But it is better for the beginner to arrange the quantities in columns, all like quantities being in the same column, thus:—

$$\begin{array}{r}
13a - 5b + c - 13d \\
-2a + b + 4c + 2d \\
-a - b + c - d \\
\hline
10a - 5b + 6c - 12d
\end{array}$$

EXAMPLE 3.

$$\begin{array}{r}
a^2bc - 7ab^2c - 6abc^2 \\
-3a^2bc + 6ab^2c + 2abc^2 \\
9a^2bc - 12ab^2c + abc^2 \\
-2a^2bc - ab^2c + 3abc^2 \\
\hline
5a^2bc - 14ab^2c
\end{array}$$

EXAMPLE 4.

$$\begin{array}{rrrr}
-6x^3 + & 3ax^2 + & 13a^2x + & a^3 \\
8x^3 & & & -8a^3 \\
+ & 6ax^2 - & 6a^2x & \\
-x^3 - & 12ax^2 - & 4a^2x + & 6a^3 \\
\hline
x^3 - & 3ax^2 + & 3a^2x - & a^3
\end{array}$$

Exercises II.

Add together:—

1. $a + 6b,\ 4a - 9b,\ 2a + 3b,\ a - b.$
2. $3ab - 4bc,\ 5ab - 5bc,\ -6ab + 8bc,\ ab + 2bc.$
3. $x + y + z,\ x + y - z,\ x + z - y,\ y + z - x.$
4. $a^2 + b^2 + c^2,\ 3a^2 - 2b^2 + 3c^2,\ 4a^2 + 3b^2 - 4c^2.$
5. $6x + 7y - 4z + 2,\ 2x - 3y + 5z - 6,\ -5x + 7y + 4z + 4.$
6. $4x + y - 5z - 3,\ 3x - 2y + z + 5,\ x + y + 3z - 6.$
7. $a^2 + ab + b^2,\ 2a^2 - 2ab + b^2,\ 3a^2 + ab - 3b^2,\ -5a^2 + 2ab + 2b^2.$
8. $x^3 - 3ax^2 + 4a^2x - 5a^3,\ 4x^3 + 4ax^2 + 5a^2x + 4a^3,\ 2x^3 - 14ax^2 + 6a^2x - a^3.$
9. $a^2b - 8ab^2 + ac^2 - 4a^2c,\ 6ab^2 - 3a^2b + 5ac^2 + 6a^2c,\ 3a^2b + 2ab^2 - 3ac^2 - 2a^2c.$
10. $3m^3 - 2m^2n - 2mn^2 + 3n^3,\ -m^3 - 2n^3,\ -m^2n + mn^2,\ -m^3 + mn^2,\ 3m^2n - 2n^3.$
11. $x^2 + 2xy + 2xz - 2yz,\ y^2 - xy + yz,\ z^2 - 3xy - 4xz + 3yz,\ +x^2 + y^2 - z^2 - 2xy + 2xz - 2yz.$
12. $2x^2 - 3z^4 + 5xz^2 + yz^2,\ x^2 + y^4 + z^4 + 2xy^2 - 2xz^2 - 2yz^2,\ x^2 - 2y^4 - 2xy^2,\ 3y^4 + 2z^4 - 2xz^2 + 5yz^2.$
13. $3a^2x^2 + 3abxy,\ 3b^2y^2 - 4abxy,\ a^2x^2 - 4b^2y^2,\ b^2y^2 - 4a^2x^2 + abxy.$
14. $a^5 - 4a^4b + a^3b^2 + 3a^2b^3 - 4ab^4 + 3b^5,\ 3a^4b - a^2b^3,\ a^5 + a^3b^2 + 3ab^4,\ 3a^5 + a^3b^2 - 3a^2b^3 + ab^4,\ a^4b - a^2b^3 + 2b^5.$
15. $7a^2 + 3ab + 4b^2,\ a^2 - ab,\ -4a^2 + 5b^2,\ ab - b^2,\ 2a^2 + 4ab - 6b^2,\ a^2 - b^2.$
16. $4x^2y - 5xy^2,\ x^2y + 7xy^2,\ -2x^2y + xy^2,\ 3x^2y - 4xy^2,\ x^2y + xy^2.$
17. $a^3 - 3a^2b + 3ab^2 - b^3,\ 2a^3 - 2b^3,\ -3a^2b + b^3,\ 4a^3 + ab^2,\ a^2b - 2ab^2.$
18. $3x^3 - 4x^2 - x + 5,\ x^3 - 2x^2 + 7x - 1,\ x^3 - x^2 - 3x - 2,\ 2x^3 - 2x^2 + 2x - 6.$
19. $2a + 3b - 4c,\ 3b + 4c - 2a,\ 4c - 2a + 3b,\ a - c,\ a + b,\ b - a + c.$
20. $p^2 - pq + 3q^2,\ 1 + 3pq,\ 2p^2 - 5,\ q^2 - 5pq,\ 3q^2 + 9,\ 4 - 5p^2.$
21. $a + b + c + 1,\ 2a - 2c,\ 3b - c + 2,\ a - b + 1,\ 4a - 2b,\ b - a - 2.$
22. $2a - 3b + 4c - 5d,\ b - 2c + 3d - a,\ c + d + 3a - 2b,\ d - a + 2b - 3c.$
23. $-x^3 + 2x^2 + x - 3,\ x^2 - 2x + 1,\ 2x^3 + 3x - 4,\ x^2 - 5,\ 2x^2 - 3x.$

24. $x^3+2x^2y+2xy^2+y^3$, x^3+y^3, x^3-3x^2y, $x^2y+xy^2-y^3$, x^3-y^3.
25. $7x+5y-8z$, $x+12y-5z$, $9y+6z-2x$, $4z-x-y$, $x-y-3z$.
26. $3a^2-4b^2+5c^2$, $a^2-b^2+c^2+1$, b^2-2c^2+3, $c^2-6+a^2+2b^2$, $3-2a^2-5c^2$.
27. $a^2-b^2+2bc-c^2$, $a^2-2ab+b^2-c^2$, $b^2-a^2-2ac-c^2$, $a^2+b^2+c^2+2ab+2ac+2bc$.
28. $3-2a-4b^2$, $a+b^2-5$, b^2+7, $1-4a$, $3b^2-5a+6$, $a-b^2+1$.
29. $4ab^2c^3-9a^2b^3c+a^3bc^2$, $a^3bc^2-ab^2c^3+3a^2b^3c$, $a^2b^3c+a^3bc^2-ab^2c^3$.
30. $2x^4-3x^2+2$, x^3-4x^2-5x, $2x^4-6x-3$, x^3+2x^2+1, x^3-2x^2+3x.

SUBTRACTION.

If b is to be taken from a the result is written $a-b$, and if $b+c$ is to be taken from a the result will evidently be less than in the last case by c, so that the result must be written $a-b-c$. Again, if $b-c$ is to be taken from a the result will be *greater* by c than that of the first case, because the quantity to be subtracted is *less* by c; the result is therefore written $a-b+c$. We therefore have the following rule for the subtraction of algebraical quantities:—

RULE.—*To subtract one algebraical expression from another, change the signs of all the terms in the expression to be subtracted, and then proceed as in Addition.*

EXAMPLE 1.—From $5a+6b-7c$ take $4a-5b+6c$. Changing the signs of the terms in the second expression it will stand thus: $-4a+5b-6c$. Adding this to the first expression we get—

$$5a+6b-7c-4a+5b-6c=a+11b-13c.$$

After the student has a little practice there will generally be *no need to write down the change of signs;* he may *imagine* that done.

EXAMPLE 2.

$$\begin{array}{lrrr} \text{From} & 3x^2 & -\ 2y & +\ 3z \\ \text{take} & 2x^2 & -\ y & +\ 2z \\ \hline & x^2 & -\ y & +\ z \end{array}$$

EXAMPLE 3.

$$\begin{array}{rrrrr} -\ a^4 & -\ a^3b & +\ 3a^2b^2 & -\ ab^3 & +\ 2b^4 \\ -\ 2a^4 & & +\ 2a\,b^2 & & +\ b^4 \\ \hline a^4 & -\ a^3b & +\ a^2b^2 & -\ ab^3 & +\ b^4 \end{array}$$

EXAMPLE 4.

$$\begin{array}{lrrrr} \text{From} & x^4 & & -\ x^2y^2 & +\ y^4 \\ \text{take} & & x^3 & -\ x^2y^2 + y^3 & \\ \hline & x^4 & -\ x^3 & -\ y^3 & +\ y^4 \end{array}$$

EXAMPLE 5.

$$\begin{array}{rrrrr} 2a^4 & -\ 3a^3 & +\ 4a^2 & -\ 5a & -\ 6 \\ a^4 & -\ 2a^3 & +\ 3a^2 & -\ 4a & -\ 4 \\ \hline a^4 & -\ a^3 & +\ a^2 & -\ a & -\ 2 \end{array}$$

Exercises III.

1. Take $16-2a$ from $18+4a$.
2. From $a+b$ take $a-b$.
3. From $a-b$ take $a+b$.

4. From $10a-5b+3c$ take $6a-b-c$.
5. From $x+2y+4z$ take $-3x+y-z$.
6. From $18ax-ky+2cz$ take $12ax+5ky-4cz$.
7. From $l+2m+3n-p$ take $l-m+4n-2p$.
8. From $x^2+2xy+y^2$ take $x^2-2xy+y^2$.
9. Take a^2+b^2+2ab from a^2-b^2-2ab.
10. From x^3-3x^2+3x-1 take x^3+3x^2+3x+1.
11. From $5x^4-12x^2y^2+y^4$ take $x^4-8x^2y^2+6y^4$.
12. Take $x^4+x^3y+x^2y^2+xy^3+y^4$ from $x^4-4x^3y+6x^2y^2-4xy^3+y^4$
13. From $a+b$ take $c-b$.
14. From $a-d$ take $b+c$.
15. From $a+2b-3c$ take $b+2c-d$.
16. From $x^2+y^2+z^2$ take $x^2-2xy+y^2-2yz+2xz$.
17. (a) From $24x^4+13a^2x^2-11a^4$ take $8x^4-3ax^3+2a^2x^2$,
(b) and from the remainder take $10x^4-3ax^3+13a^2x^2+16a^4$.
18. (a) From $15y^4-17y^2x+22x^2$ take $x^4+18x^2-14xy^2$, and
(b) take this remainder from $x^4+13y^4+2x^2-y^2x$.
19. From $3p^2-2pq+q^2$ take $3pq-2q^2+p^2$.
20. ″ a^2+5a-3 ″ a^2-a-1.
21. ″ $x-y-z$ ″ $-x-y+z$.
22. Take a^3+3a^2+3a+1 from a^3-3a^2+3a-1.
23. ″ $7a-8b+3c-4d$ ″ $a-b+5c-6d$.
24. ″ a^2+x^2 ″ $a^2-2ax+x^2$.
25. ″ x^4+x^2+1 ″ x^3+x^2+x+1.
26. ″ $3a-4b+5c+6d+e$ ″ $a-2b+6c-8d$.
27. ″ $2a^2x^2-5ax+1$ ″ $4a^2x^2+ax-6$.
28. ″ $x^3-4x^2y+9xy^2+y^3$ ″ $6x^3+3x^2y+7xy^2-2y^3$.
29. What must be added to $x-y$ to give x?
30. What must be added to $a-b+c$ to give $a+b$?

MULTIPLICATION.

$a\times b$ means the sum of b taken a times.

Thus, $a\times b$ means $b+b+b+\ldots\ldots(a$ times), and is written ab.

Or, $a\times b$ means the sum of a taken b times.

Thus, $a\times b$ means $a+a+a+\ldots\ldots(b$ times), and is written ab.

$a\times b=ab$.

$a\times -b$ means the subtraction of the sum of b taken a times.

Thus, $a\times -b$ means the subtraction of ab.

The subtraction of ab is represented by $-ab$.

$\therefore a\times -b=-ab$.

$-a\times b$ means the subtraction of the sum of a taken b times.

Thus, $-a\times b$ means the subtraction of ab.

But the subtraction of ab is represented by $-ab$.

$\therefore -a\times b=-ab$.

$-a\times -b$ means the subtraction of the sum of $-a$ taken b times.

But the sum of $-a$ taken b times is $-ab$.

Thus, $-a\times -b$ means the subtraction of $-ab$.

But the subtraction of $-ab$ is represented by $+ab$.

$\therefore -a \times -b = +ab$ or ab.

From the above we get the following general results:—

$$+a \times +b = +ab \qquad -a \times -b = +ab$$
$$+a \times -b = -ab \qquad -a \times +b = -ab$$

And from these we have the following rules:—If two quantities with the same sign are multiplied together, the result is positive. If two quantities with opposite signs are multiplied together, the result is negative. Or, stated briefly, **like signs give plus, unlike signs give minus.**

$a^m \times a^n = a \times a \times a \ldots\ldots (m$ factors$) \times a \times a \times a \ldots\ldots (n$ factors$) = a \times a \times a \times a \ldots\ldots (m+n)$ factors $= a^{m+n}$.

Hence the rule.—When **like quantities** are multiplied together **add the indices.** Thus, $a^2 \times a^3 = a^5$.

RULES FOR MULTIPLICATION.

CASE I.—When multiplier and multiplicand are both simple expressions.

(1.) Determine the sign of the product by combining the signs of the multiplier and multiplicand. Like signs give $+$, unlike signs give $-$.

(2.) Determine the numerical coefficient of the product by multiplying together the numerical coefficients of multiplier and multiplicand.

(3.) Arrange as a continued product this numerical coefficient and the various literal factors of multiplier and multiplicand.

(4.) If in multiplier and multiplicand we have different powers of the same letter, form the product of these powers according to the law $a^m \times a^n = a^{m+n}$.

(5.) The order of the factors in the product is immaterial, *e.g.* ab may be written ba; xy^2z may be written zy^2x, or xzy^2, or y^2zx, or y^2xz, or zxy^2.

EXAMPLE. $5ab^2x^3y \times 3a^2b^3x = 15a^3b^5x^4y$.

CASE II.—Where the multiplicand is a compound expression and the multiplier a simple expression.

Multiply each term of the multiplicand by the multiplier, and find the algebraical sum of the several products.

NOTE.—If different powers of the same letter occur in the terms of this sum, arrange the terms according to the descending or ascending powers of that letter.

EXAMPLE.—Multiply $3p^4q^5 + q^4r^5 - 6p^2r^4$ by $-2p^2q^3r^4$.

$$\begin{array}{l} -3p^4q^5 + q^4r^5 - 6p^2r^4 \\ \underline{-2p^2q^3r^4} \\ \hline 6p^6q^8r^4 - 2p^2q^7r^9 + 12p^4q^3r^8 \end{array}$$

CASE III.—When both multiplier and multiplicand are compound expressions.

Multiply each term of the multiplicand by each term of the mul-

tiplier, and find the algebraic sum of the several products. If like quantities occur among the products arrange them in columns, as in addition.

EXAMPLE.—Multiply $2x^2-6-4x+x^3$ by $2x+x^2-3$ (arrange according to descending powers of x).

$$\begin{array}{rrrrrr}
x^3 & +2x^2 & -4x & -6 & & \\
x^2 & +2x & -3 & & & \\
\hline
x^5 & +2x^4 & -4x^3 & -6x^2 & & \\
 & +2x^4 & +4x^3 & -8x^2 & -12x & \\
 & & -3x^3 & -6x^2 & +12x & +18 \\
\hline
x^5 & +4x^4 & -3x^3 & -20x^2 & & +18
\end{array}$$

Exercises IV.

Multiply—

		by
1.	$-10xy^2z^3$	$-2xy^4z$.
2.	$4p^2q^3r^4$	$9ax$.
3.	$-6x^3y$	$-5xy^3$.
4.	$9a^3b^4$	$-3b^2c^4d$.
5.	$-12a^2y^7$	$-ab^2cx^6$.
6.	a^2-ax+x^2	a^2.
7.	$9a^2-7-3b^2$	$6x^2$.
8.	$a^2-2ab+b^2$	$-ab$.
9.	$-3p^2q+3pq^2-q^3$	$-2pq$.
10.	$6m^2-5m-6$	$3mn$.
11.	$-7a^3+2a^2b-ac$	$-5ab^2c^3$.
12.	$10x^5-4x^3y^3-7y^5$	$2x^2y^2$.
13.	ax^3-bx^2+3x	$-c$.
14.	$p^2y^2-2pqy+q^2$	$4ax$.
15.	$-1-3mx+n^2x^2$	$-3mn$.

Multiply—

		by
16.	a^2b+4ab^2	$3a^2b-ab^2$.
17.	$3a^3b^2c-6a^2b$	$2a^3b^2c-8a^2b$.
18.	$8x^3y-2x^2y^2$	$5x^3y+3x^2y^2$.
19.	$4x^2-7$	$x+3$.
20.	$3a^2+5b^2$	$a-b$.
21.	a^2-ab+b^2	$a+b$.
22.	a^2+ab+b^2	$a-b$.
23.	x^2-2x+1	$x+1$.
24.	$x^2-3x-10$	$x-2$.
25.	$2x^2-3x+7$	$4x+5$.
26.	$3x^2+4x-5$	$2x-3$.
27.	$1+5x+6x^2$	$1+4x$.
28.	$1-9x+20x^2$	$1-3x$.
29.	$x^2+2xy+y^2$	$x+y$.
30.	$x^2-2xy+y^2$	$x-y$.

		by
31.	$5x^2-4xy+3y^2$	$2x-y$.
32.	$3a^2-5abc+2b^2c^2$	$a-4bc$.
33.	$m^4+3m^2n^2-4n^4$	$3m^2-n^2$.
34.	$a+b+c$	$a-b$.
35.	$3x-2y-z$	$3x+z$.
36.	$2p^2-4q+5r$	$3p+q$.
37.	$a^2x^2-axy+3y^2$	$y-ax$.
38.	x^3+2x^2+4x+8	$x-2$.
39.	x^3-5x^2-x+2	$3x-5$.
40.	$2x^3-x^2+6x-3$	$4x+2$.
41.	$a^3+3a^2b+3ab^2+b^3$	$a-b$.
42.	b^3-3b^2+3b-1	$b+1$.
43.	$x^3-2x^2y+2xy^2-y^3$	$2x-3y$.
44.	$a^2+2ab+b^2-c^2$	$a-b$.
45.	$a^2-b^2+2bc-c^2$	$b+c$.
46.	$2x^3+4x^2+8x+16$	$3x-6$.
47.	$3a^6-6a^4x+12a^2x^2-24x^3$	$2a^2+4x$.

Multiply—

48. $a^4-a^3+a^2-a+1$ by a^2+2a+1.
49. $x^4+x^2y^2+y^4-x^3y-xy^3$ x^2+xy-y^2.
50. $a^2+b^2+c^2-ab-ac-bc$ $a+b+c$.
51. $ab+ac+bc$ $a-b+c$.
52. $x^4+2ax^3+3a^2x^2+2a^3x+a^4$ $x^2-2ax+a^2$.
53. $x^2+2xy-5xz+4y^2+10yz+25z^2$ $x-2y+5z$.
54. $1+2x+3x^2+4x^3+5x^4$ $1-2x+x^2$.
55. $a^2-2ab-3ac+4b^2-6bc+9c^2$ $a+2b+3c$.
56. $a^4-4a^3+6a^2-4a+1$ a^3-3a^2+3a-1.
57. $1+x+x^2+x^3+x^4+x^5$ $1-x+x^2-x^3+x^4-x^5$.

Find the continued product of—

58. $a+1$, $a-1$, a^2+1, a^4+1.
59. $ab+x$, $ab-x$, $a^2b^2+x^2$, $a^4b^4+x^4$.
60. $x+y$, x^2-xy+y^2, $x-y$, x^2+xy+y^2.
61. a^2-ab+b^2, a^2+ab+b^2, $a^4-a^2b^2+b^4$.
62. $1-xy$, $1-xz$, $1-yz$.

DIVISION.

Since $a\times b=ab$, $ab\div a=b$.
$a\times -b=-ab$, $-ab\div a=-b$.
$-a\times b=-ab$, $-ab\div -a=b$.
$-a\times -b=ab$, $ab\div -a=-b$.

Hence in division as in multiplication the rule of signs is—**like signs give plus, unlike minus.**

Since $4a\times 3b=12ab$, $12ab\div 4a=3b$.

Hence the rule of coefficients:—

Divide the coefficient of the dividend by that of the divisor

Since $a^3\times a^4=a^7$, $a^7\div a^3=a^4$.

Hence the rule of indices:—

Subtract the indices of like letters.

Since $b^2\div b^2=b^0$, and also $=1$, $b^0=1$.

CASE I.—To divide one simple expression by another. Apply in order the rules for signs, coefficients, and indices.

EXAMPLE 1. $12xy\div 4=3xy$.

EXAMPLE 2. $-2abx\div x=-2ab$.

EXAMPLE 3. $8a^7b^4\div -2a^3b^2=-4a^4b^2$.

EXAMPLE 4. $-30a^4b^4x^4y^4\div -5a^2x^3=6a^2b^4xy^4$.

CASE II.—To divide a compound expression by a simple one.

Divide each term of the dividend separately by the divisor as in Case I.

EXAMPLE 1.

$$2\,)\,\underline{4a+6b}$$
$$2a+3b$$

EXAMPLE 2.

$$b\,)\,\underline{ab-b^2}$$
$$a-b$$

EXAMPLE 3.

$$-3abc\,)\,\underline{-3a^2bc^3+6a^3bc^2}$$
$$ac^2-2a^2c$$

EXAMPLE 4.

$$-4a^2b^2y\,)\,\underline{8a^7b^5x^3y-12a^4b^2xy^3}$$
$$-2a^5b^3x^3+3a^2xy^2$$

Case III.—To divide one compound expression by another.

1. **Arrange** as far as possible both divisor and dividend according to ascending or descending powers of some common letter.
2. **Divide** the first term of the dividend by the first term of the divisor for the first term of the quotient.
3. **Multiply** each term of the divisor by the term thus found.
4. **Subtract** the product from the dividend.
5. Consider the remainder a new dividend, and proceed as before.

Example 1.—Divide $x^2-9x+20$ by $x-4$.

$$
\begin{array}{l}
x-4\,)\,x^2-9x+20\,(\,x-5 \\
\qquad\quad x^2-4x \\
\qquad\qquad -5x+20 \\
\qquad\qquad -5x+20
\end{array}
$$

Example 2.—Divide $a^4+a^2b^2+b^4$ by a^2+ab+b^2.

$$
\begin{array}{l}
a^2+ab+b^2\,)\,a^4+a^2b^2+b^4\,(\,a^2-ab+b^2 \\
\qquad a^4+a^3b+a^2b^2 \\
\qquad\quad -a^3b \\
\qquad\quad -a^3b-a^2b^2-ab^3 \\
\qquad\qquad +a^2b^2+ab^3+b^4 \\
\qquad\qquad +a^2b^2+ab^3+b^4
\end{array}
$$

Example 3.—Divide $x^4-7x+90-30x^2$ by $9+x^2-7x$.

Arrange divisor and dividend according to descending powers of x.

$$
\begin{array}{l}
x^2-7x+9\,)\,x^4-30x^2-7x+90\,(\,x^2+7x+10 \\
\qquad x^4-7x^3+9x^2 \\
\qquad\quad 7x^3-39x^2-7x \\
\qquad\quad 7x^3-49x^2+63x \\
\qquad\qquad 10x^2-70x+90 \\
\qquad\qquad 10x^2-70x+90
\end{array}
$$

Example 4.—Divide $x^3-y^3+z^3+3xyz$ by $x-y+z$.

$$
\begin{array}{l}
x-y+z\,)\,x^3+3xyz-y^3+z^3\,(\,x^2+xy-xz+y^2+yz+z^2 \\
\qquad x^3-x^2y+x^2z \\
\qquad\quad x^2y-x^2z+3xyz \\
\qquad\quad x^2y-xy^2+xyz \\
\qquad\qquad -x^2z+xy^2+2xyz \\
\qquad\qquad -x^2z+xyz-xz^2 \\
\qquad\qquad\quad xy^2+xyz+xz^2-y^3 \\
\qquad\qquad\quad xy^2-y^3+y^2z \\
\qquad\qquad\qquad xyz+xz^2-y^2z \\
\qquad\qquad\qquad xyz-y^2z+yz^2 \\
\qquad\qquad\qquad\quad xz^2-yz^2+z^3 \\
\qquad\qquad\qquad\quad xz^2-yz^2+z^3
\end{array}
$$

Exercises V.

Divide—

1. $3x^2y^2z - x^2yz^2 - 4xy^2z^2$ by $-xyz$.
2. $8a^5b - 20a^4b^2 + 12a^3b^3 + 16a^2b^4$ by $4a^2b$.
3. $a^2b - 3abc + 2ab^2cd$ by ab.
4. $12a^2xy - 9ax^2y + 15ab^2x - 3a^3cx^2y$ by $3ax$.
5. $-20a^5bc^4 + 15a^4b^2c^3d - 15abc + 10a^3b^2c$ by $-5abc$.
6. $a^3b^4x^3y^4 - 2a^4b^3x^2y^3 - 3a^5b^2xy^2 + 4a^6by$ by $-a^3by$.
7. $4x - 12x^2y + 8x^3y^2z - 16ax^4y^3z^2$ by $4x$.
8. $a^3b^3d - ab^3cd^2 + 2a^2d^5 - 2abcd$ by $-ad$.
9. $-2a^7 + 10a^5 + 18a^3 - 4a$ by $-2a$.
10. $-6a^6x^8 + 9a^7x^{10} - 12a^8x^{12} + 15a^9x^{14}$ by $-3a^5x^6$.

Divide—

11. $x^2+13x+42$ by $x+6$.
12. $x^2+27x+180$ by $x+15$.
13. $x^2-13x+36$ by $x-4$.
14. $x^2-33x+272$ by $x-17$.
15. $a^2+3a-54$ by $a+9$.
16. $a^2+11a-312$ by $a-13$.
17. a^2-a-12 by $a+3$.
18. $a^2-7a-98$ by $a-14$.
19. y^2-64 by $y+8$.
20. $4y^2-121$ by $2y-11$.

Divide—

21. $6x^2+5x-6$ by $2x+3$.
22. $5a^2-18a-8$ by $a-4$.
23. $8x^2+24x+10$ by $2x+5$.
24. $28x^2-2x-6$ by $4x-2$.
25. $30x^2-x-1$ by $1+6x$.
26. $9a^2-27a+20$ by $3a-4$.
27. $12b^2c^2-58bc-10$ by $bc-5$.
28. y^4+7y^2+12 by y^2+3.
29. x^4-x^2-156 by x^2-13.
30. $9x^4-81$ by $3x^2+9$.

Divide—

31. $12x^3 - x^2y + 4y^3$ by $3x+2y$.
32. $b^3 + 13b^2 + 52b + 60$ by $b^2+8b+12$.
33. $x^3 + 12x^2y + 39xy^2 + 28y^3$ by $x^2+5xy+4y^2$.
34. $a^3 + 9a^2b - 10ab^2 - 168b^3$ by $a^2+3ab-28b^2$.
35. $12p^3 - 11p^2 + 9p + 18$ by $3p^2-5p+6$.
36. $x^3 - x^2 - 14x + 24$ by x^2+2x-8.
37. $x^3 - 9x^2y + 26xy^2 - 24y^3$ by $x^2-5xy+6y^2$.
38. $6a^3b^3 - 8a^2b^2 - 56ab - 32$ by $3a^2b^2-10ab-8$.
39. $x^3 - 61x + 180$ by $x^2+4x-45$.
40. $x^3 - 9x^2y + 108y^3$ by $x^2-3xy-18y^2$.
41. $6x^2 + 4xy - 5xz - 16y^2 + 40yz - 25z^2$ by $2x+4y-5z$.
42. $9a^2 - 6ab + 24ac - 8b^2 - 2bc + 15c^2$ by $3a-4b+5c$.
43. $a^2 + 2ab + b^2 - c^2$ by $a+b+c$.
44. $m^2 - n^2 + 2nx - x^2$ by $m-n+x$.
45. $a^2x^2 - 2abxy + b^2y^2 - 1$ by $ax-by-1$.
46. $4x^2 - 12xy + 9y^2 - 16z^2$ by $2x-3y+4z$.
47. $9a^2 - 4b^2 + 16bc - 16c^2$ by $3a+2b-4c$.
48. $a^2 - b^2 + x^2 - y^2 - 2ax - 2by$ by $a-b-x-y$.
49. $a^3 + b^3 - 6b^2 + 12b - 8$ by $a+b-2$.
50. $x^3 - y^3 + 3y^2z - 3yz^2 + z^3$ by $x^2+xy-xz+y^2-2yz+z^2$.

BRACKETS.

$$
\begin{aligned}
\text{Since } a+(b+c)&=a+b+c,\\
a+(b-c)&=a+b-c,\\
a-(b+c)&=a-b-c,\\
\text{and } a-(b-c)&=a-b+c.
\end{aligned}
$$

we have the following

Rules for the Removal of Brackets.

(1.) If the bracketed quantity be preceded by the sign +, simply remove the brackets.

(2.) If the bracketed quantity be preceded by the sign −, remove the brackets and change the sign of every term within them.

(3.) When there are brackets within brackets, first clear away the innermost, then the next, and so on.

A straight line, called a **vinculum** (band or bond), placed over several terms, may be used instead of brackets.

Example 1.—Simplify by clearing away brackets and collecting like terms, $2a+3b-\{a+4b-(5a+2b)\}$.

$$
\begin{aligned}
&2a+3b-\{a+4b-(5a+2b)\}\\
&=2a+3b-\{a+4b-5a-2b\}\\
&=2a+3b-a-4b+5a+2b=6a+b.
\end{aligned}
$$

Example 2.—Simplify $4x-[6x+3y-\{x-5y+z+(7x-6y+4z-\overline{2x-y-8z})\}]$

$$
\begin{aligned}
&4x-[6x+3y-\{x-5y+z+(7x-6y+4z-\overline{2x-y-8z})\}]\\
&=4x-[6x+3y-\{x-5y+z+(7x-6y+4z-2x+y+8z)\}]\\
&=4x-[6x+3y-\{x-5y+z+7x-6y+4z-2x+y+8z\}]\\
&=4x-[6x+3y-x+5y-z-7x+6y-4z+2x-y-8z]\\
&=4x-6x-3y+x-5y+z+7x-6y+4z-2x+y+8z\\
&=4x-13y+13z.
\end{aligned}
$$

Example 3.—Simplify $4a-7[a+2\{3a-5(a-1)\}-8]$

$$
\begin{aligned}
&4a-7[a+2\{3a-5(a-1)\}-8]\\
&=4a-7[a+2\{3a-5a+5\}-8]\\
&=4a-7[a+2\{-2a+5\}-8]\\
&=4a-7[a-4a+10-8]\\
&=4a-7[-3a+2]\\
&=4a+21a-14\\
&=25a-14.
\end{aligned}
$$

Exercises VI.

Simplify—

1. $4a+(3a-6)$.
2. $6-(4a+3)$.
3. $5x^2-\overline{3x-7}$.
4. $2x+3+(6x-9)$.
5. $7x+5y-(x+5y)$.
6. $a+8b-(a-8b)$.
7. $4a+3b+2c+(a+2b+3c)$.
8. $9a-5b+c-(6a+3b+5c)$.
9. $2x+4y-z-(x-3y-9z)$.
10. $a^2+2ab+b^2+(a^2-2ab+b^2)$.
11. $a^2+2ab+b^2-(a^2-2ab+b^2)$.
12. $x-(4x+3)+(6x+2)$.
13. $5a+b-(4a+\overline{3b-c})$.

14. $4a^2-(2a^2+a-\overline{4a+1})$.
15. $a-\{b-(c-d)\}$.
16. $a^3-\{3a^2b+(3ab^2-b^3)\}$.
17. $7x^2-\{4x^2-(3x^2-5)\}$.
18. $2x-(3x-5y)-\{x+7y-(8x+y)\}$.
19. $6x^3-\{3x^3+6x^2-(3x^2+6x-\overline{3x+6})\}$.
20. $3a-(2b+3c)-\{c-2b+(a-\overline{b+2c})\}$.
21. $3-\{3+(2a-1)\}-\{4a-(5+a)\}$.
22. $4-(1-x)+\{3-(5x+\overline{x-7})\}$.
23. $13a-[11a+\{9a-(7a+5a)\}]$.
24. $x^3-(x^2+x)-[1-x+\{x^2-(x^3+x^2-1)\}]$.
25. $13a+7b-[c+6a-\{4b-3c+(2a-\overline{b+9c})\}]$.
26. $5-[5-\{5-(5-\overline{5-a})\}]$.
27. $3a-[7b+\{2c-(a-9b)\}]+[4a-\{6c+(5a-b)\}]$.
28. $x^2-[x^2-\{x^2-(x^2+y^2)\}]-[y^2-\{y^2-(y^2-\overline{y^2+x^2})\}]$.
29. $1-[1-x-\{1-x-x^2-(1-x-x^2-x^3-\overline{1-x-x^2-x^3-x^4})\}]$.
30. $40-10\{4-(2+3)\}$.
31. $\frac{1}{2}-\frac{1}{4}(\frac{1}{3}-\frac{1}{2})-\frac{1}{3}\{\frac{1}{8}-3(2-\overline{1+\frac{1}{2}})-6\}$.
32. $4a-3[\{2b-(1-a)-b\}+2(1-a)]$.
33. $x\left[-x-\frac{1}{x}\left\{x+x\left(1-\overline{\frac{1}{x}+x}\right)\right\}+1\right]$.
34. $4b^2(3a^2+3ab+b^2)^2-\{(a+b)^3-(a-b)^3\}^2$.
35. $\{(x+y+z)^2-(x-\overline{y+z})^2\}^2-16x^2(y+z)^2$.

SIMPLE EQUATIONS.

Two expressions connected by the sign of equality $(=)$ constitute either an **equation** or an **identity**.

In an **equation** the statement of equality is only true when the letters in it have a particular value or values. Thus the statement $x+2=8-x$ is only true when x has the particular value 3. $x+2=8-x$ is therefore an equation.

In an **identity** the statement of equality is true whatever be the values given to the letters in it. Thus the statement $(a+b)^2=a^2+2ab+b^2$ is true whatever be the numerical value of a or b. $(a+b)^2=a^2+2ab+b^2$ is therefore an identity. Two numerical expressions which are equal to one another and are connected by the sign $=$ also constitute an identity, as $8\times5=6\times7-2$. The following are also examples of identities:—

$$(x+y)\,(x-y)=x^2-y^2$$
$$(x+2)\,(x^2-2x+1)=x^3-3x+2$$
$$\frac{x+a}{x-a}-\frac{x-a}{x+a}=\frac{4ax}{x^2-a^2}$$

The expressions on each side of the sign $=$ are called the **sides** or **members** of the equation, that on the left-hand side being called the **first** side and that on the right the **second** side.

A letter in an equation which must have a particular value or

values before the statement of equality can be true is called an **unknown quantity**. Unknown quantities are usually denoted by the letters at the end of the alphabet, as x, y, z.

The **root** of an equation is the particular value of the unknown quantity which, when substituted for it, reduces the equation to an identity. Thus in the equation $x+5=2x+3$ it will be found that the root is 2; putting this for x in the equation we get $2+5=2\times2+3$, which is an identity. The root of an equation is also said to **satisfy** the equation.

The process of finding the root or roots of an equation is called **solving** the equation.

When, in an equation which is in its simplest form, the highest power of the unknown quantity is the first, the equation is said to be of the **first degree** or a **simple equation**. And when the highest power of the unknown quantity is the second the equation is of the **second degree** or a **quadratic equation**. In like manner we have *cubic equations* or equations of the **third degree**, and so on. Of the following equations (1) is of the first degree or a simple equation, (2) is of the second degree or a quadratic equation, (3) is of the third degree or a cubic equation, and (4) is of the fourth degree or a biquadratic equation.

$$2(x-3)+5=3(x-2) \qquad (1)$$
$$x^2-5x=6 \qquad (2)$$
$$x^3+2x-7=0 \qquad (3)$$
$$x^4-3x^3-7x^2-27x=18 \qquad (4)$$

An equation of the first degree is also said to be of **one dimension**; one of the second degree of **two dimensions**, and so on.

An equation with one unknown quantity has as many roots as it has dimensions.

The equations in this chapter are solved by the application of one or more of the following axioms:—

(1) If equals be added to equals, the results are equal.
(2) If equals be subtracted from equals, the results are equal.
(3) If equals be multiplied by equals, the results are equal.
(4) If equals be divided by equals, the results are equal.

From the first two axioms we derive the following important proposition, namely—*Any term may be transferred from one side of an equation to the other by changing its sign.*

For example, let $x+b=a-x$.
Add x to each side; then by axiom (1)

$$x+x+b=a-x+x,$$

that is $$2x+b=a.$$

subtract b from each side; then by axiom (2)

$$2x+b-b=a-b,$$

that is $$2x=a-b.$$

Thus we have taken the $-x$ from the right to the left hand side of the equation by changing it to $+x$. Also we have taken the $+b$

from the left to the right hand side by changing it to $-b$. From this it is manifest that every term of an equation may have its sign changed without destroying the equality expressed by the equation.

From axioms (3) and (4) we derive the following:—*If every term on each side of an equation be multiplied or divided by the same quantity, the results are equal.*

The principle just stated is most frequently applied to clear an equation of fractions.

We will now apply the foregoing principles to the solution of a few examples.

EXAMPLE 1.—Solve the equation $5x-7=2x+8$.

$$5x-2x=8+7;$$

that is $$3x=15;$$

divide by 3; then $$x=\frac{15}{3}=5.$$

If 5 be put for x in the original equation we find that each side is equal to 18; therefore 5 is the root of the equation.

EXAMPLE 2.—Solve $3(x+1)-7=2(3-x)$.

Multiplying and removing the brackets,

$$3x+3-7=6-2x;$$

transposing $$3x+2x=6+7-3;$$

that is $$5x=10;$$

therefore $$x=\frac{10}{5}=2.$$

EXAMPLE 3.—Solve $5(x+3)-4(x-3)+3x=3(x+2)-2(x-15)$.

Multiplying and removing the brackets we get

$$5x+15-4x+12+3x=3x+6-2x+30;$$

transposing $$5x-4x+3x-3x+2x=6+30-15-12;$$

collecting the terms $$3x=9;$$

therefore $$x=\frac{9}{3}=3.$$

EXAMPLE 4.—Solve $\cdot3(x-\cdot5)-\cdot25(x-2\cdot8)=\cdot2(x+8)-\cdot4(x-\cdot5)$.

Multiplying and removing the brackets we get

$$\cdot3x-\cdot15-\cdot25x+\cdot7=\cdot2x+1\cdot6-\cdot4x+\cdot2;$$

transposing $$\cdot3x-\cdot25x-\cdot2x+\cdot4x=1\cdot6+\cdot2+\cdot15-\cdot7;$$

collecting the terms $$\cdot25x=1\cdot25;$$

therefore $$x=\frac{1\cdot25}{\cdot25}=5.$$

EXAMPLE 5.—Solve $\frac{3x}{2}+\frac{x}{3}-\frac{x+2}{4}-\frac{x-5}{6}=6.$

The first step will be to clear the equation of fractions. This is done by multiplying both sides by 12 (the L.C.M. of the denominators of the fractions).

Thus $\frac{12\times 3x}{2}+\frac{12x}{3}-\frac{12(x+2)}{4}-\frac{12(x-5)}{6}=12\times 6;$

that is $18x+4x-3x-6-2x+10=72;$

collecting the terms $17x=68;$

therefore $x=\frac{68}{17}=4.$

EXAMPLE 6.—Solve $\frac{1}{2}(x+\frac{1}{3})+\frac{1}{3}(x+\frac{1}{4})=\frac{1}{4}(x+\frac{1}{5})+\frac{1}{5}(x+\frac{1}{6}).$
Multiplying both sides by 60 (the L.C.M. of 2, 3, 4, and 5) we get

$$30(x+\tfrac{1}{3})+20(x+\tfrac{1}{4})=15(x+\tfrac{1}{5})+12(x+\tfrac{1}{6});$$

multiplying out and removing brackets the above becomes

$$30x+10+20x+5=15x+3+12x+2;$$

transposing $30x+20x-15x-12x=3+2-10-5;$

collecting the terms $23x=-10;$

therefore $x=-\frac{10}{23}.$

We may now give the rules for solving a simple equation of one unknown quantity.

(1) *Clear the equation, if necessary, of fractions.* This is done by multiplying every term by the L.C.M. of all the denominators.

(2) *Transpose all the terms which contain the unknown quantity to one side of the equation, and the known quantities to the other.*

(3) *Collect the terms.*

(4) *Divide both sides by the complete coefficient of the unknown quantity.*

Exercises VII.

Solve the equations—

1. $13x=182.$
2. $15x=225.$
3. $4x+5=21.$
4. $2x+5x+3x=50.$
5. $13x-17x-2x+8x=6.$
6. $3x+5=x+9.$
7. $14x-11=10x+5.$
8. $24x-27=13x-5.$
9. $2x+3=8x+45.$
10. $ax+c=bx+d.$
11. $6(x-5)+4=5(x-4).$
12. $9(x-11)=6(x+2)-12.$
13. $3(x-4)+5(x-6)-6=0.$
14. $29x-6(x+15)-71=0.$
15. $3(x+7)+6x=8-5x.$
16. $3(x+7)+7(x-5)=4(x+2)+2(x+1).$
17. $\frac{x}{3}+\frac{x}{4}=14.$
18. $\frac{3x}{4}-\frac{x}{2}+\frac{x}{5}=9.$
19. $\frac{x}{5}-1=\frac{6}{5}-\frac{x}{6}.$
20. $\frac{x}{6}-\frac{x}{3}-\frac{2}{21}=\frac{x}{7}+\frac{4}{15}-\frac{2x}{5}.$
21. $\frac{x}{2}-\frac{x}{4}+\frac{x}{3}=21.$
22. $\frac{4x}{3}-6=\frac{3x}{4}+\frac{5}{12}.$
23. $\frac{3x-38}{4}=2-5x.$
24. $\frac{4x+3}{2}+\frac{3x+5}{3}=12\frac{1}{6}.$
25. $5(x+1)-3(x+2)=9.$
26. $\frac{x-5}{3}+\frac{x}{2}=12-\frac{x-10}{3}.$
27. $\frac{1}{3}(5x-1)-6(22-3x)=2x-3.$
28. $\frac{2x}{3}-4+\frac{x}{4}=\frac{7x}{12}-2.$

29. $15x-3(x+1)=4x-25.$

30. $35(2x-1)-25(x+1)=4-5x.$

31. $5(2x-3)-12(3x+1)=5(x-2)-11(x-3).$

32. $35x-55(x+6)+205(x-27)=75x-459.$

33. $\frac{34x-56}{15}-\frac{7x-3}{5}=\frac{7x-5}{3}+2\frac{1}{6}.$

34. $\frac{17x-5}{18}-\frac{3x-25}{9}=\frac{7-5x}{4}-4\frac{17}{36}.$

35. $\frac{3x-7}{5}-\frac{3x+7}{4}=\frac{5x-9}{8}-\frac{3x-9}{6}.$

36. $x-\frac{3}{5}-\frac{5(x-2)}{4}=\frac{3}{2}(x-\frac{1}{10}).$

37. $\frac{1}{3}(x-7\frac{1}{2})+\frac{4}{5}(x-15)=\frac{1}{2}(4x-11).$

38. $\frac{7x-28}{3}-3\frac{1}{4}+\frac{4x-21}{7}=x-7\frac{1}{4}-\frac{9-7x}{8}.$

39. $1+\frac{x}{2}-\frac{2x}{3}=\frac{3x}{4}-4\frac{1}{2}.$

40. $\frac{x+1}{6}-\frac{2x-6}{7}=6-\frac{4x-13\frac{2}{3}}{14}.$

41. $x-6-\frac{x-12}{3}=\frac{x-4}{2}+\frac{x-8}{4}.$

42. $\frac{x}{2}+\frac{1+x}{3}+\frac{2+x}{4}+\frac{3+x}{5}=4.$

43. $\frac{3-x}{2}-\frac{1}{3}\left(\frac{3-2x}{4}\right)=\frac{4(2x+3)}{7}+\frac{1}{4}-\frac{3x+1}{2}.$

44. $x-\frac{x+2}{5}+\frac{x-6}{11}-\frac{x-4}{7}=0.$

45. $3x+\frac{7x-4}{4}+2(3x-7)=28.$

46. $\frac{9x+7}{2}-\left(x-\frac{x-2}{7}\right)=36.$

47. $\frac{3x+7}{14}-\frac{2x-7}{21}+2\frac{3}{4}=\frac{x-4}{4}.$

48. $\frac{4x-21}{9}+3\frac{3}{4}+\frac{57-3x}{4}=241-\frac{5x-96}{12}-11x.$

49. $\frac{4x-21}{7}+7\frac{3}{4}+\frac{7x-28}{3}=x+3\frac{3}{4}-\frac{9-7x}{8}.$

50. $3\frac{2}{3}-x-\frac{9x}{2}+8=\frac{3}{2}x-17-\frac{3x}{5}.$

51. $\frac{x}{3}+\frac{2x+8\frac{1}{2}}{9}-\frac{13x-2}{36}=\frac{7x}{12}-\frac{x+16}{36}.$

52. $7\left(x+\frac{1}{3}\right)-5x\left(\frac{1}{3x}+\frac{1}{2\frac{1}{2}}\right)=4.$

PROBLEMS PRODUCING SIMPLE EQUATIONS.

One of the uses of algebra lies in the application of the equational form to the solution of problems.

For example, *to find the number, the half of which added to* 3, *is equal to* 12 *diminished by* $\frac{1}{4}$*th that number.*

To arrive at the required number by ordinary arithmetic would be somewhat difficult, and would require close connected reasoning. Whereas, if we represent the required number by x and express

algebraically, according to the terms of the problem, the relation between x and the known numbers, we have an equation,

$$\frac{x}{2}+3=12-\frac{x}{4},$$

where all the facts are stated in a connected and tangible form, and the value of x is determined by solving the equation.

$$\text{Thus } \frac{x}{2}+\frac{x}{4}=9.$$
$$2x+x=36.$$
$$3x=36.$$
$$\therefore x=12.$$

As another example of questions presenting difficulty of solution by ordinary arithmetical methods consider the following:—

EXAMPLE 2.—*The ages of a father and his son together are* 80 *years. If the age of the son be doubled it will exceed the father's age by* 10 *years. What are their respective ages?*

Representing the father's age by x, the son's age will be $(80-x)$; and double the son's age will be $2(80-x)$.

Thus we form the equation—

$$2(80-x)=x+10.$$
That is $160-2x=x+10.$
Or $-3x=-150.$
Dividing by -3, $x=50$, the father's age.
$80-50=30$, the son's age.

The following problems and their solutions will further illustrate this method:—

EXAMPLE 3.—*A's age is thrice that of B, and B's age is as much below* 38 *as A's age is above it. Required the age of each.*

Let $x=$ B's age.
Then $3x=$ A's age.
$38-x=$ the number of years B's age is below 38.
$3x-38=$ the number of years A's age is above 38.
And according to the terms of the problem,

$$38-x=3x-38.$$
$$-4x=-76.$$
$$x=19=\text{B's age}.$$
$$3x=57=\text{A's age}.$$

EXAMPLE 4.—*A class is composed of three times as many boys as girls. One day, when four boys and four girls were absent, it was found that there were present four times as many boys as girls. What is the number in the class?*

In such a case as this proceed as follows:—

Let $x=$ number of girls.
$\therefore 3x=$ number of boys.
$x-4=$ number of girls present when 4 are absent.
$3x-4=$ number of boys present when 4 are absent.

Now according to the terms of the problem,

$$3x-4=4 \text{ times } (x-4)=4(x-4).$$

That is $3x-4=4x-16.$

$$-x=-12.$$

$$x=12=\text{number of girls in class.}$$

$$\therefore\ 3x=36=\text{number of boys in class.}$$

$$\therefore\quad 48=\text{number in class.}$$

EXAMPLE 5.—*A man walking at the rate of four miles an hour, covers a certain distance in three hours less than another who walks at the rate of three miles an hour. Find the distance.*

Let $x=$ the distance in miles.

$\frac{x}{4}=$ the time in hours of the one.

$\frac{x}{3}=$ the time in hours of the other.

According to the terms of the problem,

$$\frac{x}{4}=\frac{x}{3}-3.$$

$$\therefore\ 3x=4x-36.$$

$$-x=-36.$$

$$\therefore\ x=36 \text{ miles.}$$

Exercises VIII.

1. Twice a certain number is greater than 10 by 4. What is the number?
2. One number exceeds another by 3, and their sum is 25. Find the numbers.
3. Divide 36 into two such parts that the one may exceed the other by 4.
4. The sum of two numbers is 24. If to one of them five times the other be added the sum is 40. Find the numbers.
5. Divide 340 into two such parts that one part may be sixteen times the other.
6. The sum of two numbers is 50 and their difference is three times the less. Required the numbers.
7. The sum of three numbers, each of which is one-third of the number following, is 117. Find the numbers.
8. Find the number whose half diminished by 1 is equal to its third part.
9. Find the number whose half diminished by 3 is equal to its eighth part multiplied by 3.
10. Find a number, the sum of whose sixth and eighth parts is less than its half by 10.
11. A fish weighs 10 lbs. and two-thirds of its weight more. Find the weight of the fish.
12. What was the wealth of a man, who, when he gave away the third and fifth parts of it and £350 more, had nothing left?

13. A boy had four times as many red marbles as white ones. He gave away 12 red ones for 8 white ones, and then had three times as many red as white. How many of each had he?
14. A vessel belonging to three persons A, B, and C, was lost. A possessed a third of the vessel, B a half of what A possessed, and C the remainder. C's loss was £1000. What was the value of the vessel?
15. A boy playing marbles gains four times as many as he started with, and then loses 5. He now finds that he has 95 left. How many had he at first?
16. A number of apples are divided equally among 6 boys. If there had been 7 boys each would have got 3 apples less. What was the number of apples?
17. One cistern can hold twice as much as another. The larger one when one-third filled contains 10 gallons more than the smaller when half full. How many gallons can each hold?
18. The amount of leakage from a cistern is 21 gallons less than a third of the whole contents; 63 gallons now remain. What was the leakage?
19. A prize of £4400 is divided among two officers, four men, and six boys. Each officer receives twice as much and each boy half as much as a man. What does each receive?
20. A farmer sells 3 horses, 5 cows, and 30 sheep for £600. The price of a sheep is one-third that of a cow. The price of a horse is five times that of a sheep. Find the price of each.
21. A man walks a certain distance in nine hours. If his rate had been half-a-mile more per hour, his time would have been one hour less. How far did he walk?
22. How many eggs at 3 for twopence should be purchased in order that 10 shillings may be gained on retailing them at 4 for threepence?
23. A can dig a garden in 10 days, B in 12 days, and C in 15 days. In what time can they do it working together?
24. A boy plucks from a tree a certain number of apples. A second takes four for every three the first takes. Together these two have five times as many as a third takes; and the three have 42. How many does each take?

SUPPLEMENT TO MULTIPLICATION.

By actual multiplication we get—

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$
$$(x+a)(x-b)=x^2+(a-b)x-ab,$$
$$(x-a)(x+b)=x^2-(a-b)x-ab,$$
and $$(x-a)(x-b)=x^2-(a+b)x+ab,$$

the product of expressions of these forms may be put down without going through the ordinary process of multiplication.

Note—

(1.) That the product consists of three terms in descending powers of x.

(2.) That the first term of the product is x^2.

(3.) That the coefficient of the second term is the **sum** of the second terms of the factors.

(4.) That the third term is the **product** of the second terms of the factors.

EXAMPLE 1.—What is the product of $x+7$ and $x+5$?

The product consists of three terms.

The first term is x^2.

The coefficient of the second term is $7+5$, that is 12.

The third term is 7×5, that is 35.

Therefore the product is $x^2+12x+35$.

EXAMPLE 2.—What is the product of $x-7$ and $x+5$?

As before the product consists of three terms of which the first is x^2.

The coefficient of the second term is $-7+5$, that is -2.

The third term is -7×5, that is -35.

Therefore the product is $x^2-2x-35$.

By actual multiplication we find that the continued product of $x+a$, $x+b$, $x+c$ is $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$, from which we observe that—

(1.) The product consists of four terms in descending powers of x, beginning with x^3.

(2.) The coefficient of the second term is the **sum** of the second terms of the factors.

(3.) The coefficient of the third term is the **sum** of the **products** of each pair of the second terms of the factors.

(4.) The fourth term is the **product** of the second terms of the factors.

EXAMPLE.—Find the continued product of $x-3$, $x+4$, and $x-5$.

The first term is x^3.

The coefficient of the second term is $-3+4-5$, that is -4.

The coefficient of the third term is $(-3\times 4)+(-3\times -5)+(4\times -5)$, that is $-12+15-20$, that is -17.

The fourth term is $-3\times 4\times -5$, that is 60.

Therefore the product is $x^3-4x^2-17x+60$.

By actual multiplication—

$$(1.)\qquad (a+b)^2=a^2+2ab+b^2.$$
$$(2.)\qquad (a-b)^2=a^2-2ab+b^2.$$
$$(3.)\qquad (a+b)(a-b)=a^2-b^2.$$

Hence—

I. The square of the **sum** of any two numbers is equal to the sum of the squares of these numbers **increased** by twice their product.

II. The square of the **difference** of any two numbers is equal to the sum of the squares of these numbers **diminished** by twice their product.

III. **The product of the sum and difference of any two numbers is equal to the difference of their squares.**

These theorems should be carefully studied and committed to memory.

By Theorem I. the square of the sum of any two numbers may be found.

$$\begin{aligned}(2x+3y)^2 &= (2x)^2+2\,.\,2x\,.\,3y+(3y)^2\\ &= 4x^2+12xy+9y^2.\\ 21^2 &= (20+1)^2\\ &= 20^2+2\,.\,20\,.\,1+1^2\\ &= 400+40+1\\ &= 441.\end{aligned}$$

By Theorem II. the square of the difference of any two numbers may be found.

$$\begin{aligned}(3a-5b)^2 &= (3a)^2-2\,.\,3a\,.\,5b+(5b)^2\\ &= 9a^2-30ab+25b^2.\\ 29^2 &= (30-1)^2\\ &= 30^2-2\,.\,30\,.\,1+1^2\\ &= 900-60+1\\ &= 841.\end{aligned}$$

By Theorem III. the product of the sum and difference of any two numbers may be found.

$$\begin{aligned}(4a+7b)(4a-7b) &= (4a)^2-(7b)^2\\ &= 16a^2-49b^2.\\ 42\times 38 &= (40+2)(40-2)\\ &= 40^2-2^2\\ &= 1600-4\\ &= 1596.\end{aligned}$$

These results may be applied to expressions of more than two terms by enclosing two or more of the terms in brackets and treating them as one.

$$\begin{aligned}(a+b+c)^2 &= \{(a+b)+c\}^2\\ &= (a+b)^2+2(a+b)c+c^2\\ &= a^2+2ab+b^2+2ac+2bc+c^2.\\ (x^2-3xy+2y^2)^2 &= \{x^2-(3xy-2y^2)\}^2\\ &= (x^2)^2-2\,.\,x^2(3xy-2y^2)+(3xy-2y^2)^2\\ &= x^4-6x^3y+4x^2y^2+9x^2y^2-12xy^3+4y^4\\ &= x^4-6x^3y+13x^2y^2-12xy^3+4y^4.\\ (x^2+xy+y^2)(x^2-xy+y^2) &= \{(x^2+y^2)+xy\}\{(x^2+y^2)-xy\}\\ &= (x^2+y^2)^2-(xy)^2 \text{ by Theor. III.}\\ &= x^4+2x^2y^2+y^4-x^2y^2\\ &= x^4+x^2y^2+y^4.\end{aligned}$$

The square of an expression of any number of terms may be found by the following rule:—

Take the square of each term, and twice the product of each term into all those that follow.

Find the square of x^3-2x^2-3x+4.

the square of the first term is x^6
the square of the second term is $+4x^4$
the square of the third term is $+9x^2$
the square of the fourth term is $+16$
twice the product of the first term into the second is $-4x^5$
twice the product of the first term into the third is $-6x^4$
twice the product of the first term into the fourth is $+8x^3$
twice the product of the second term into the third is $+12x^3$
twice the product of the second term into the fourth is $-16x^2$
twice the product of the third term into the fourth is $-24x$.

Collect like terms and arrange according to descending powers of x.

$$(x^3-2x^2-3x-4)^2=x^6-4x^5-2x^4+20x^3-7x^2-24x+16.$$

The following example is of great importance and should be carefully studied.

Find the continued product of—

$$a+b+c,\ a+b-c,\ a-b+c,\ -a+b+c.$$

$$\begin{aligned}
&(a+b+c)\,(a+b-c)\,(a-b+c)\,(-a+b+c)\\
&=\{(a+b)+c\}\{(a+b)-c\}\{c+(a-b)\}\{c-(a-b)\}\\
&=\{(a+b)^2-c^2\}\{c^2-(a-b)^2\}\\
&=\{a^2+2ab+b^2-c^2\}\{c^2-a^2+2ab-b^2\}\\
&=\{2ab+(a^2+b^2-c^2)\}\{2ab-(a^2+b^2-c^2)\}\\
&=(2ab)^2-(a^2+b^2-c^2)^2\\
&=4a^2b^2-(a^4+b^4+c^4+2a^2b^2-2a^2c^2-2b^2c^2)\\
&=4a^2b^2-a^4-b^4-c^4-2a^2b^2+2a^2c^2+2b^2c^2\\
&=2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4.
\end{aligned}$$

Exercises IX.

Find mentally the product of—

1. $x+3$ and $x+7$.
2. $x-5$ and $x-6$.
3. $x+9$ and $x-2$.
4. $x-8$ and $x+3$.
5. $a+6$ and $a+6$.
6. $x+1$, $x+2$, and $x+3$.
7. $x-1$, $x-2$, and $x-3$.
8. $x+2$, $x-3$, and $x+4$.
9. $x-5$, $x-6$, and $x+7$.
10. $x+3$, $x+7$, and $x-10$.

Without performing the operation of multiplication expand the following:—

11. $(31)^2$, $(42)^2$, $(53)^2$, $(61)^2$, $(82)^2$, $(103)^2$.
12. 18×22, 27×33, 46×54, 69×71, 85×95, 79×81.
13. $(x+y)^2$, $(p+q)^2$, $(x+1)^2$, $(2+y)^2$.
14. $(x+3y)^2$, $(4x+y)^2$, $(3x+4y)^2$, $(5x+8y)^2$.
15. $(3a-1)^2$, $(4b-7)^2$, $(5-6c)^2$, $(1-7d)^2$.
16. $(x^2+x+1)^2$, $(x^2+2x+1)^2$, $(x^2+2x+2)^2$.
17. $(2x^2+3x+4)^2$, $(3x^2+5x+6)^2$, $(4x^2+7x+9)^2$.
18. $(x^2-2xy-y^2)^2$, $(x^2-3xy+4y^2)^2$, $(x^2+4xy-5y^2)^2$.

19. $(x+y+z)^2$, $(x-y+z)^2$, $(x-y-z)^2$.
20. $(x^3+x^2+x+1)^2$, $(x^3+2x^2+x+2)^2$.
21. $(2x^3+3x^2+4x+1)^2$, $(3x^3+5x^2+x+4)^2$.
22. $(x^3-3x^2y-4xy^2+2y^3)^2$, $(x^3-2x^2y-4xy^2+y^3)^2$.
23. $(x+y+z)(x+y-z)$, $(x+y+z)(x-y+z)$.
24. $(x^2+x+1)(x^2+x-1)$, $(x^2+x+1)(x^2-x+1)$.
25. $(a^2+ab+b^2)(a^2+ab-b^2)$, $(a^2+ab+b^2)(a^2-ab+b^2)$.
26. $(a+b+c+d)(a+b-c-d)$.
27. $(a-b+c+d)(a-b-c-d)$.
28. $(x^3+x^2+x+1)(x^3+x^2-x-1)$.
29. $(x^3-2x^2+3x+4)(x^3-2x^2-3x-4)$.
30. $(p+q+r)(p+q-r)(p-q+r)(-p+q+r)$.

FACTORS.

The general results which have been given in the supplement to multiplication, together with one or two others, are very useful in resolving expressions into factors.

Formula. $x^2+(a+b)x+ab=(x+a)(x+b)$.
EXAMPLE. $x^2+9x+20=x^2+(4+5)x+4\times 5=(x+4)(x+5)$.
Formula. $x^2+(a-b)x-ab=(x+a)(x-b)$.
EXAMPLE. $x^2+2x-15=x^2+(5-3)x-5\times 3=(x+5)(x-3)$.
Formula. $x^2-(a-b)x-ab=(x-a)(x+b)$.
EXAMPLE. $x^2-x-30=x^2-(6-5)x-6\times 5=(x-6)(x+5)$.
Formula. $x^2-(a+b)x+ab=(x-a)(x-b)$.
EXAMPLE. $x^2-11x+24=x^2-(8+3)x+8\times 3=(x-8)(x-3)$.
Formula. $a^2+2ab+b^2=(a+b)^2$.
EXAMPLE. $9x^2+12xy+4y^2=(3x)^2+2\times 3\times 2xy+(2y)^2=(3x+2y)^2$.
Formula. $a^2-2ab+b^2=(a-b)^2$.
EXAMPLE. $9-30x+25x^2=3^2-2\times 3\times 5x+(5x)^2=(3-5x)^2$.
Formula. $a^2-b^2=(a+b)(a-b)$.
EXAMPLE. $16x^2-49y^2=(4x)^2-(7y)^2=(4x+7y)(4x-7y)$.
Formula. $a^3+b^3=(a+b)(a^2-ab+b^2)$.
EXAMPLE. $27x^3+y^3=(3x)^3+y^3=(3x+y)(9x^2-3xy+y^2)$.
Formula. $a^3-b^3=(a-b)(a^2+ab+b^2)$.
EXAMPLE. $125x^6y^3-8z^6=(5x^2y)^3-(2z^2)^3=(5x^2y-2z^2)(25x^4y^2+10x^2yz^2+4z^4)$.

Exercises X.

Resolve the following expressions into factors:—

1. x^2+5x+6.
2. $x^2-8x+16$.
3. x^2+6x+5.
4. $a^2-12ab+35b^2$.
5. $x^2+8x+16$.
6. $a^2-14ab+49b^2$.
7. y^4-13y^2+40.
8. $a^4+9a^2b+14b^2$.
9. $21+10x+x^2$.
10. $36-12x^2y^2+x^4y^4$.
11. $28x^2+11xy+y^2$.
12. $(x+y)^2+11(x+y)+30$.
13. $x^2+4x-21$.
14. $x^2-3x-18$.
15. $x^2+9x-10$.
16. x^2+x-72.

17. $a^2+11ab-26b^2$.
18. $a^2-4ab-45b^2$.
19. $y^2-yz-156z^2$.
20. $x^2y^2+xy-42$.
21. $x^4+13x^2y-48y^2$.
22. $y^6-3y^3z^2-40z^4$.
23. $(a+b)^2-4(a+b)-60$.
24. $x^2+2x(y+z)-15(y+z)^2$.
25. $(x+y)^2+4(x^2-y^2)-5(x-y)^2$.
26. $144x^2y^2-z^2$.
27. $81x^4y^2-121z^6$.
28. $x^{10}-100y^{20}$.
29. $1-49x^6y^4z^2$.
30. x^2-1.
31. $(a-b)^2-c^2$.
32. $(2a+3b)^2-4c^2$.
33. $x^2-(y+z)^2$.
34. $x^2-(3y-4z)^2$.
35. $c^4-(a^2+ab-b^2)^2$.
36. $(x+y)^2-(a-b)^2$.
37. $a^2-2ab+b^2-c^2-2cd-d^2$.
38. $a^2+2ab+b^2-c^2+2cd-d^2$.
39. $1-2xy+x^2y^2-a^2b^2-2abcd-c^2d^2$.
40. $a^2-10ab+25b^2-c^2-12cd-36d^2$.
41. $(a+b)^2-(a-b)^2$.
42. $(4a+3b)^2-(4a-3b)^2$.
43. 127^2-123^2.
44. 301^2-299^2.
45. 567^2-433^2.
46. x^3y^3+64.
47. $(a+b)^3+125c^3$.
48. $8a^3+(b-c)^3$.
49. $(x+y)^3+(x-y)^3$.
50. $216+(4a-5c)^3$.
51. $(x+y)^3-z^3$.
52. $(a+b)^3-(a-b)^3$.
53. $(2a+3b)^3-(2a-3b)^3$.
54. $64-(a-x)^3$.
55. x^6-64y^6.
56. $1-a^6$.

GREATEST COMMON MEASURE.

If two or more quantities possess the same factor, that common factor is said to be a **common measure** of those quantities.

Thus, in $4a^2b$, $14abc$, ba^3c, a occurs as a factor of each quantity, b also occurs as a factor of each quantity.

a and b are therefore common measures of those three quantities.

The **greatest common measure** of two or more quantities is the greatest factor that is common to each of those quantities.

NOTE.—The term **greatest factor** does not imply that it is arithmetically the greatest, but that it is the **factor of greatest dimensions.**

CASE I.—Quantities of one term.

EXAMPLE.—Find the G.C.M. of $32a^3bc$, $16ab^2c^2$, and $24a^2bc^3$.

(1.) Determine the G.C.M. of the numerical coefficients 32, 16, 24. 8 is the G.C.M. of the numerical coefficients.

(2.) Determine the factor of highest dimensions common to a^3bc, ab^2c^2, a^2bc^3.

abc is the greatest common factor.

$\therefore 8abc$ is the G.C.M. of the 3 quantities.

CASE II.—To find the greatest common measure of compound quantities which are capable of being conveniently decomposed into factors.

Decompose each quantity into its factors and determine the greatest factor common to each.

EXAMPLE 1.—Find the G.C.M. of—

$$5x^2+15x+10,\ 2x^3+10x^2+8x, \text{ and } 7x^3+7.$$
$$5x^2+15x+10=5(x^2+3x+2)=5(x+1)(x+2).$$
$$2x^3+10x^2+8x=2x(x^2+5x+4)=2x(x+1)(x+4).$$
$$7x^3+7=7(x^3+1)=7(x+1)(x^2-x+1).$$

The only factor common to all the quantities is $(x+1)$.

$\therefore$ G.C.M. $=(x+1)$.

EXAMPLE 2.—Find the G.C.M. of—

$$10(x^4-2x^2y^2+y^4),\ 15(x^4-y^4), \text{ and } 5x^3y-5xy^3.$$
$$10(x^4-2x^2y^2+y^4)=10(x^2-y^2)(x^2-y^2).$$
$$15(x^4-y^4)=15(x^2+y^2)(x^2-y^2).$$
$$5x^3y-5xy^3=5xy(x^2-y^2).$$

The greatest factor common to the three quantities is $5(x^2-y^2)$.

$\therefore$ G.C.M. $=5(x^2-y^2)$.

EXAMPLE 3.—Find the G.C.M. of—

$$30(x^4-1),\ 12(x-1)(x^2-1)^2,\ 18(x^7-x^6+x-1).$$

Here $30(x^4-1)=30(x^2-1)(x^2+1)=5.6(x+1)(x-1)(x^2+1)$,

and $12(x-1)(x^2+1)^2=12(x-1)(x^2+1)(x^2+1)=2.6(x-1)(x^2+1)(x^2+1)$,

and $18(x^7-x^6+x-1)=18(x-1)(x^6+1)=3.6(x-1)(x^2+1)(x^4-x^2+1)$.

Examining these results we find that the greatest factor common to all the quantities is $6(x-1)(x^2+1)$.

$\therefore$ The G.C.M. of the three quantities is $6(x-1)(x^2+1)$.

CASE III.—To find the G.C.M. of two compound quantities which are not conveniently decomposed into their ultimate factors.

For example, find the G.C.M. of—

$$x^2-5x-14 \text{ and } x^3-9x^2+7x+49.$$

Here we have two expressions in which the terms are arranged according to the descending powers of x.

The G.C.M. of these quantities means the expression of the highest degree in reference to x that is a common factor of the two quantities.

Thus the **highest common algebraic divisor** of two algebraic quantities is known as their G.C.M.

In finding the G.C.M. of two arithmetical quantities such as 104 and 195, we proceed as in the margin. 13, the last divisor, is the G.C.M. of 104 and 195.

```
104 ) 195 ( 1
      104
       91 ) 104 ( 1
             91
             13 ) 91 ( 7
                  91
```

Example 1.—In finding the G.C.M. of the proposed quantities, $x^2-5x-14$ and $x^3-9x^2+7x+49$, we proceed in a similar manner, thus:—

$$
\begin{array}{r|l}
x^2-5x-14 & x^3-9x^2+7x+49\ (x-4 \\
 & \underline{x^3-5x^2-14x} \\
 & \quad -4x^2+21x+49 \\
 & \quad \underline{-4x^2+20x+56} \\
 & \qquad\qquad\quad x-7
\end{array}
$$

The first remainder is $x-7$.

$$
\begin{array}{r|l}
x-7 & x^2-5x-14\ (x+2 \\
 & \underline{x^2-7x} \\
 & \quad 2x-14 \\
 & \quad \underline{2x-14}
\end{array}
$$

The remainder is 0; and the G.C.M. of the two quantities is $(x-7)$.

Example 2.—Find the G.C.M. of $2x^3-11x^2+6x-5$ and x^3-7x^2+9x+5.

Arrange for division as before—

$$
\begin{array}{r|l}
x^3-7x^2+9x+5 & 2x^3-11x^2+6x-5\ (2 \\
 & \underline{2x^3-14x^2+18x+10} \\
 & \qquad 3x^2-12x-15
\end{array}
$$

The first remainder is $3x^2-12x-15$ or $3(x^2-4x-5)$, and as 3 is not a factor of x^3-7x^2+9x+5, we reject it from $3(x^2-4x-5)$.

Then (x^2-4x-5) now becomes the divisor, and the previous divisor becomes the dividend.

$$
\begin{array}{r|l}
x^2-4x-5 & x^3-7x^2+9x+5\ (x-3 \\
 & \underline{x^3-4x^2-5x} \\
 & \quad -3x^2+14x+5 \\
 & \quad \underline{-3x^2+12x+15} \\
 & \qquad\qquad 2x-10
\end{array}
$$

For the second remainder, $2x-10$, or $2(x-5)$, we reject the factor 2 as it is not a factor of x^2-4x-5. Thus our divisor is now $x-5$ and our dividend x^2-4x-5.

$$
\begin{array}{r|l}
x-5 & x^2-4x-5\ (x+1 \\
 & \underline{x^2-5x} \\
 & \qquad x-5 \\
 & \qquad \underline{x-5}
\end{array}
$$

The remainder is 0; and the G.C.M. is $x-5$.

Example 3.—Find the G.C.M. of $8x^4+12x^3+2x^2+2$ and $15x^4+21x^3-3x+3$.

$$8x^4+12x^3+2x^2+2=2(4x^4+6x^3+x^2+1),$$
$$\text{and } 15x^4+21x^3-3x+3=3(5x^4+7x^3-x+1).$$

As 2 and 3 have no common measure we reject them, and arrange for division.

$$4x^4+6x^3+x^2\;\;+1\;)\;5x^4+7x^3\;\;\;-x+1\;($$

To avoid fractions multiply the dividend by 4, and we have—

$$\begin{array}{r|l}
4x^4+6x^3+x^2+1\;) & 20x^4+28x^3\qquad\;\;-4x+4\;(\;5 \\
 & \underline{20x^4+30x^3+5x^2\qquad+5} \\
 & \quad\;-\;\;2x^3-5x^2-4x-1
\end{array}$$

Divide previous divisor by remainder, first changing all the signs of the latter.

$$\begin{array}{r|l}
2x^3+5x^2+4x+1\;) & 4x^4+\;\;6x^3+\;\;\;x^2\qquad+1\;(\;2x-2 \\
 & \underline{4x^4+10x^3+\;\;8x^2+2x} \\
 & \quad-\;\;4x^3-\;\;7x^2-2x+1 \\
 & \quad\underline{-\;\;4x^3-10x^2-8x-2} \\
 & \qquad\qquad\quad 3x^2+6x+3
\end{array}$$

Reject the factor 3 from the remainder, and divide as before—

$$\begin{array}{r|l}
x^2+2x+1\;) & 2x^3+5x^2+4x+1\;(\;2x+1 \\
 & \underline{2x^3+4x^2+2x} \\
 & \qquad\;\;x^2+2x+1 \\
 & \qquad\;\;\underline{x^2+2x+1}
\end{array}$$

The G.C.M. is x^2+2x+1.

The rules may then be stated thus:—

(1.) Arrange the terms of each of the compound quantities in descending powers of some letter occurring in both.

(2.) Divide that quantity which has the highest powers of this letter by the other. (If both the quantities are of the same degree, either may be made the divisor.)

(3.) If there is a remainder divide the former divisor by this remainder, and so on, dividing the previous divisor by the remainder until the remainder is nothing.

(4.) The last divisor is the G.C.M.

Additional rules.

(1.) To prevent fractions occurring, the divisor and the dividend may be multiplied by fitting numbers.

(2.) All the signs of either divisor or dividend may be changed.

(3.) Any factor occurring in divisor and not in dividend, or in dividend and not in divisor, may be rejected as not forming part of the G.C.M.

(4.) Any factor occurring in both divisor and dividend may be reserved as forming part of the G.C.M.

To show the reason of the above rules—

Let A and B represent the quantities whose G.C.M. is to be found; and let the operation be represented as in the margin. D is the last divisor.

$$\begin{array}{l}
\mathrm{A}\;)\;\mathrm{B}\;(\;a \\
\quad\;\underline{a\mathrm{A}} \\
\quad\;\;\mathrm{C}\;)\;\mathrm{A}\;(\;b \\
\qquad\quad\;\;\underline{b\mathrm{C}} \\
\qquad\quad\;\;\;\mathrm{D}\;)\;\mathrm{C}\;(\;c \\
\qquad\qquad\quad\;\;\underline{c\mathrm{D}} \\
\qquad\qquad\qquad 0
\end{array}$$

I. To show that D is a **common measure** of A and B.

D measures C, since $C=cD$.

And since $A=bC+D$.

$\therefore\ A=b\times cD+D=(bc+1)D$.

$\therefore$ D measures A.

And since $B=aA+C$.

$\therefore\ B=a(bcD+D)+cD=\{a(bc+1)+c\}D$.

$\therefore$ D measures B.

Therefore D is a **common measure** of A and B.

II. Next to show that D is the **greatest common measure** of A and B.

(*a*) Whatever measures A and B measures also C.

Let m be a measure of A and B, and let $A=sm$, and
$B=tm$.

Now since $C=B-aA$.

$\therefore\ C=tm-asm=m(t-as)$.

Therefore m is a measure of C.

(*b*) And by similar reasoning, whatever measures A and C measures D.

(*c*) Therefore whatever measures A and B measures D.

(*d*) But the greatest measure of D is D. Therefore the **greatest common measure** of A and B is D.

Corollary.—Seeing that whatever measures A and B measures also the remainder C, and also measures C multiplied by any quantity, therefore we may multiply the remainder by any quantity, provided that that quantity is not at the same time a factor of the divisor.

To find the G.C.M. of three compound quantities not readily decomposed into factors.

(1.) Find the G.C.M. of any two of those quantities.

(2.) The G.C.M. of this result from (1.) and the third quantity is the G.C.M. of the three quantities.

Illustration.—Let A, B, C represent the three quantities.

(1.) Let the G.C.M. of A and B be D.

Let $A=aD$. Let $B=bD$.

$\therefore$ a and b have **no common factor.**

(2.) Let the G.C.M. of D and C be m.

Let $D=dm$. Let $C=cm$.

$\therefore$ d and c have **no common measure.**

$A=aD=adm$. $B=bD=bdm$. $C=cm$.

Since a and b have no common factor, and d and c no common factor,

$\therefore$ the G.C.M. of A, B, and C is m.

To find the G.C.M. of more than three compound quantities proceed similarly.

Exercises XI.

Find the greatest common measure in the following examples:—

1. $2a^3x^3y^2$, $14ax^2y^3$.
2. $21a^5b^2c^4$, $15a^3bc^2$.
3. $45a^2b^2c^4x^3$, $30a^4b^5c^6x^2$.
4. $18a^{12}b^6c^4x^2y^5$, $81a^6c^3x^3y^3$.
5. $15x^2y^3z^5$, $20x^2y^2z+35xy^3z^2$.
6. $33a^7b^2c^3$, $44a^4b^6c^4-22a^2b^4c$.
7. $3a^2(b^2-c^2)$, $12(b+c)$.
8. $4ab(x^2+2xy+y^2)$, $20(x+y)$.
9. $34(x+a)^3$, $51(x^2-a^2)$.
10. $xy^2(x^2-4)$, $x^2y(x+2)$.
11. x^3-a^3, $x^2-2ax+a^2$.
12. x^3+27, x^2+6x+9.
13. $x^2+3x-28$, x^2-x-12.
14. $x^2+6x-16$, $x^2+13x+40$.
15. $x^2+7x+12$, $x^2-2x-15$.
16. $4x^2+20x+21$, $4x^2-4x-15$.
17. $a^2x^2-5ax-6$, $a^2x^2-6ax-7$.
18. $x^2+5ax+6a^2$, $x^2+2ax-3a^2$.
19. $m^2-mn-2n^2$, $3m^2-4mn-4n^2$.
20. $4x^2+2x-30$, $6x^2-13x-5$.
21. $6x^2+13x+6$, $2x^2+11x+12$.
22. $15x^2-16x-7$, $10x^2-29x+21$.
23. $3x^3-12x^2+x-4$, $6x^3+15x^2+2x+5$.
24. x^3+8x^2+8x+7, x^3-6x^2-6x-7.
25. $x^3-8x^2+14x-12$, x^3+2x^2-6x+8.
26. $2x^3-11x^2-20x-7$, $3x^3-16x^2-33x-14$.
27. $6x^2+2x+13$, $12x^3-26x^2+16x-65$.
28. $2x^4+9x^2+5x+12$, $2x^4+4x^3+13x^2+11x+12$.
29. $2x^4-6x^3+15x^2-21x+28$, $3x^4-9x^3+7x^2+15x-20$.
30. $2x^4+9x^3+9x^2+9x+7$, $2x^4+7x^3+2x^2+5x-7$.
31. $45x^3y+3x^2y^2-9xy^3+6y^4$, $54x^2y-24y^3$.
32. $18x^3+51x^2+44x+12$, $3x^3-x^2+x+2$, $21x^2-13x-18$.
33. x^3-2x-1, $2x^3+x^2-5x-3$, and x^3-9x^2+7x+8.
34. $x^3+6x^2+11x+6$, $x^3+9x^2+26x+24$, and $2x^3+9x^2+7x-6$.
35. $x^4+5x^3+5x^2-5x-6$, $2x^4+11x^3+10x^2-11x-12$, and $5x^4-x^2-4$.
36. $2x^3-25x^2-48x-9$, $10x^3+79x^2+166x+105$, and $2x^4+11x^3+10x^2-11x-12$.

LEAST COMMON MULTIPLE.

When one quantity is exactly divisible by another the former is said to be a **Multiple** of the latter.

When one quantity is exactly divisible by two or more others it is said to be a **Common Multiple** of those quantities.

Thus 12 is a common multiple of 2, 3, and 4.
$6a^3b^2$ „ „ of $3a^2$, $2b$, and a^3b^2.
a^4-b^4 „ „ of $a+b$, a^2-b^2, a^2+b^2.

Of all the common multiples of two or more algebraical expressions, the one which has the lowest power of the letter involved is called the Least Common Multiple.

Case I.—To find the L.C.M. of two or more simple expressions: multiply together the L.C.M. of the coefficients and the highest powers of all the letters that occur.

EXAMPLE.—Find the L.C.M. of $8a^4b^3c^2d$, $12a^5b^2c^4d$, and $32a^3b^4ce^5$.

The L.C.M. of 8, 12, and 32 is 96.

The highest power of a is a^5, of b is b^4, of c is c^4, of d is d, of e is e^5.

Therefore the L.C.M. is $96a^5b^4c^4de^5$.

CASE II.—To find the L.C.M. of compound quantities easily decomposed into their elementary factors. Resolve the given quantities into their elementary factors, and treat these factors as letters are treated in Case I.

EXAMPLE 1.—Find the L.C.M. of x^2-y^2, and of $3x+3y$.

$$x^2-y^2=(x+y)(x-y)$$
$$3x+3y=3(x+y).$$

The factors 3, $x+y$, and $x-y$ occur in their first power only, and there are no other factors. Therefore $3(x+y)(x-y)$ is the L.C.M.

EXAMPLE 2.—Find the L.C.M. of $(x^2-1)^2$, $2x^2-4x+2$, and x^2-1.

$$(x^2-1)^2=(x^2-1)(x^2-1)=(x+1)(x-1)(x+1)(x-1).$$
$$=(x+1)^2(x-1)^2$$
$$2x^2-4x+2=2(x^2-2x+1)=2(x-1)^2$$
$$x^2-1 \qquad =(x+1)(x-1).$$

Therefore $2(x+1)^2(x-1)^2$ is the L.C.M. required.

CASE III.—To find the L.C.M. of two compound quantities not easily decomposed into their elementary factors.

EXAMPLE.—Find the L.C.M. of—

$$x^3-x^2-24x-36 \text{ and } x^3-7x^2-21x+27.$$

First—Determine the G.C.M. of the two expressions.

$$x+3 \text{ is the G.C.M.}$$

Second—Determine the co-factor of $(x+3)$ in each of the expressions.

The co-factor of $x+3$ in $x^3-x^2-24x-36$ is $x^2-4x-12$.

The co-factor of $x+3$ in $x^3-7x^2-21x+27$ is $x^2-10x+9$.

i.e. $x^3-x^2-24x-36=(x+3)(x^2-4x-12)$.

$x^3-7x^2-21x+27=(x+3)(x^2-10x+9)$.

And since $x+3$ is the G.C.M. of the two expressions, $x^2-4x-12$ and $x^2-10x+9$ have no common measure.

The L.C.M. is therefore $(x+3)(x^2-10x+9)(x^2-4x-12)$.

Thus the L.C.M. of two expressions is the continued product of the G.C.M. of the expressions and its co-factors in the expressions. Now the product of the G.C.M. and either of the co-factors, will be one of the given expressions. This product is to be multiplied by the co-factor of the G.C.M. in the other expressions. Hence the

RULE:—

The L.C.M. of two expressions is the product of either expression and the co-factor of the G.C.M. in the other expression.

This rule may be represented thus:—

Let A and B represent the two expressions of which the L.C.M. is required, and let M be the G.C.M. of A and B.

Let $A=Ma$. $\therefore$ a is the co-factor of M in A.
Let $B=Mb$. $\therefore$ b is the co-factor of M in B.
$\because$ M is the G.C.M. of A and B,
$\therefore$ a and b have **no common measure.**
$\therefore$ the L.C.M. is Mab.
Now $Mab=(Ma)b=Ab$,
or $Mab=(Mb)a=Ba$.

NOTE.—Since $Mab=\dfrac{Ma\times Mb}{M}=\dfrac{AB}{M}$,
therefore the L.C.M. may also be found by dividing the product of the two expressions by the G.C.M.

To find the L.C.M. of three quantities.

Let A, B, C be the quantities.
Let N = L.C.M. of A and B.
Let M = L.C.M. of N and C.
Then M = L.C.M. of A, B, C.

Since every common multiple of A and B is a multiple of N, therefore every common multiple of A, B, and C is a multiple of N and C. Also every multiple of N and C is a multiple of A, B, and C. Consequently the L.C.M. of N and C is the L.C.M. of A, B, C.

Thus M is the L.C.M. of A, B, C.

Exercises XII.

Find the L.C.M. in each of the following:—

1. $a^2b^2,\ 5ac^2,\ 2a^3b^3c^3$.
2. $6a^2,\ 4b^2,\ 9c^2$.
3. $20abx,\ 4a^2x^3,\ 10b^2x^2$.
4. $9b^4y^5,\ 21a^2xy,\ 14b^3y^6$.
5. $12,\ 3a^2,\ 6bc,\ 5c^2$.
6. $14ac^2z,\ 21bx^3,\ 4a^3y^2,\ 3b^4z^2$.
7. $2x^3by^2,\ 3ab^2z,\ 4x^4cz^2,\ 5a^2y^4$.
8. $x^2-5x+6,\ x^2-9$.
9. $x^3-y^3,\ x-y$.
10. $x^2+2xy+y^2,\ x^3+y^3$.
11. $ab,\ a^2+ab,\ ab+b^2$.
12. $4a^2-4b^2,\ 6ab,\ a^3-b^3$.
13. $9-x^2,\ 3-x,\ 3+x$.
14. $x^2+y^2,\ x^3y-xy^3,\ 2x^4-2y^4$.
15. $a+b,\ b+c,\ c+a$.
16. $x^2-6x+9,\ x^2+6x+9,\ x^2-9$.
17. $x^2-6x+5,\ x^2-3x+2,\ x^2-8x+12$.
18. $2x^3-16,\ 3x^2-12x+12,\ x^2+2x+4$.
19. $x+x^2,\ 1-x^2,\ 1+2x+x^2$.
20. $x^2+2x-3,\ x^2+5x+6,\ x^2+x-2$.
21. $x^3-4x^2+6x-4,\ x^3-3x^2+3x-2$.
22. $x^3-3x^2y+4xy^2-2y^3,\ x^3-x^2y-2xy^2+2y^3$.
23. $x^3+4x^2-5x,\ x^3-6x+5$.
24. $x^3+4x^2-5,\ x^3-3x+2$.
25. $4x^2-12x+9,\ 8x^3-27$.
26. $x^3+3x^2y+4xy^2+12y^3,\ x^3+4x^2y+4xy^2+3y^3$.
27. $x^3+x^2y-11xy^2+10y^3,\ x^3-4x^2y+2xy^2+4y^3$.

28. $x^3+x^2-10x+8,\ x^3+9x^2+17x-12.$
29. $x^3-4x^2+2x+3,\ x^3-x^2-7x+3.$
30. $4x^2-8x-60,\ x^3-2x^2-18x-9.$
31. $3x^3-9x^2-12x,\ x^4-x^3-5x^2+x+4.$
32. $x^3+3x^2y+xy^2-2y^3,\ x^4-4x^2y^2-xy^3+2y^4.$
33. $x^4-2x^3-13x^2+22x-8,\ x^4-6x^3+12x^2-19x+12.$
34. $x^4-1,\ x^4+x^3+x^2+x-4.$
35. $x^4-2x^3y+7x^2y^2-4xy^3+10y^4,\ x^4-2x^3y+2x^2y^2+6xy^3-15y^4.$
36. $x^3+7x^2+8x-16,\ x^3-13x+12,\ 3x^3+13x^2-16.$
37. $x^4-1,\ x^3+2x^2+x+2,\ x^3-3x^2+x-3.$
38. $x^3-4x^2+5x-2,\ x^3-6x^2+11x-6,\ x^3+8x^2-4x-4.$
39. $x^4-y^4,\ x^4-4x^2y^2+3y^4,\ 2x^3-4x^2y-2xy^2+4y^3.$
40. $2(x^3+y^3),\ 3(x^4-y^4),\ 4(x^5+y^5).$

FRACTIONS.

An algebraic fraction is defined and treated in the same way as an arithmetical fraction.

If an apple is divided into 3 equal parts, and a boy gets 2 of these equal parts, he gets two-thirds of the apple. His share is written thus: $\frac{2}{3}$.

$\frac{2}{3}$ is a **fraction**, 2 is the **numerator** and 3 is the **denominator**.

The denominator tells the number of parts into which unity is divided.

The numerator tells the number of such parts taken.

If an apple were divided into b equal parts, and a boy got a of those parts, his share would be written thus: $\frac{a}{b}$.

$\frac{a}{b}$ is a fraction. Unity is divided into b equal parts, and a of those parts are taken.

Again, if the apple had been divided into 10 times as many parts, and the boy had got 10 times as many of those parts, his share would have been $\frac{10a}{10b}$; but this quantity is the same as $\frac{a}{b}$, because, although the parts are 10 times as small, he gets 10 times as many of them.

The numerator here was multiplied by 10, but at the same time the denominator also was multiplied by 10, and the fraction is unchanged in value.

In the same way $\frac{ma}{mb}$ is the same as $\frac{a}{b}$. For in the new fraction $\frac{ma}{mb}$, unity is divided into m times as many parts, and each part is thus m times as small; but there are m times as many of these diminished parts taken, and therefore the quantity represented by $\frac{ma}{mb}$ is the same as that represented by $\frac{a}{b}$.

LAW.—If the numerator and denominator of a fraction be multiplied by the same quantity, the value of the fraction is unchanged. For example—

$$\frac{a}{b}=\frac{ma}{mb}.$$

It follows immediately from this law that if the numerator and denominator be divided by the same quantity the value of the fraction is unchanged.

$$\text{For } \frac{ma\div m}{mb\div m}=\frac{a}{b}=\frac{ma}{mb}.$$

A fraction is **simplified** by dividing its numerator and denominator by a factor common to both.

EXAMPLE.—Simplify the fraction $\frac{ab}{ca}$.

Here a is a factor common to both numerator and denominator. Divide both by a and we have—

$$\frac{ab}{ca}=\frac{b}{c}.$$

A fraction is in its simplest form when the numerator and denominator contain no common factor.

To convert a fraction into its simplest form.

Divide the numerator and denominator by the greatest factor common to both, that is, by their G.C.M.

EXAMPLE 1.—Reduce to its simplest form—

$$\frac{14a^2xy}{21aby^2}.$$

Here the G.C.M. of numerator and denominator is $7ay$. Divide both by $7ay$, and the simplified fraction is—

$$\frac{2ax}{3by}.$$

EXAMPLE 2.—Simplify $\frac{x^3-x^2+x-1}{x^6+1}$.

Decomposing numerator and denominator into factors, we have—

$$\frac{x^2(x-1)+(x-1)}{(x^4-x^2+1)(x^2+1)}=\frac{(x-1)(x^2+1)}{(x^4-x^2+1)(x^2+1)}=\frac{x-1}{x^4-x^2+1}.$$

EXAMPLE 3.—Simplify $\frac{x^3+15x^2+74x+120}{x^3-61x-180}$.

Neither numerator nor denominator is readily decomposed into factors.

Find the G.C.M. of numerator and denominator.

The G.C.M is $x^2+9x+20$.

$$\frac{x^3+15x^2+74x+120}{x^3-61x-180}=\frac{(x^2+9x+20)(x+6)}{(x^2+9x+20)(x-9)}=\frac{x+6}{x-9}.$$

Exercises XIII.

1. $\dfrac{18a^4bc^2x^3y^5}{27ab^4d^2y^3}$.

2. $\dfrac{4a^2b-2ab^2}{6a^2b^2}$.

3. $\dfrac{9xyz}{3x^2y-6y^2z}$.

4. $\dfrac{4ax^2-8bx^2}{3ay-6by}$.

5. $\dfrac{4a^4b^2c-2a^3b^4c^2}{6a^2bc^3+8ab^3c^4}$.

6. $\dfrac{8ax-12ay}{6bx-9by}$.

7. $\dfrac{a^2-b^2}{4a^2+4ab}$.

8. $\dfrac{3x^2+6xy}{x^2-4y^2}$.

9. $\dfrac{x^2-5x+4}{3x^2-12x}$.

10. $\dfrac{5x+10}{x^2-2x-8}$.

11. $\dfrac{a^3-8b^3}{3a^3-6a^2b}$.

12. $\dfrac{a+b}{a^2-b^2}$.

13. $\dfrac{a-b}{a^3-b^3}$.

14. $\dfrac{x^3+8y^3}{x^2+4xy+4y^2}$.

15. $\dfrac{x^2+7x+12}{x^2+10x+21}$.

16. $\dfrac{x-5}{x^2-10x+25}$.

17. $\dfrac{a^2-9b^2}{a^3-27b^3}$.

18. $\dfrac{a^2+15ab+36b^2}{a^2+12ab+27b^2}$.

19. $\dfrac{a^2-1}{a^2+6a+5}$.

20. $\dfrac{a^2+2ab+b^2-c^2}{a^2-b^2+2bc-c^2}$.

21. $\dfrac{a+b+c}{a^2-b^2-2bc-c^2}$.

22. $\dfrac{1-2x+x^2-y^2}{1-x^2+2xy-y^2}$.

23. $\dfrac{x^2+ax+bx+ab}{x^2-ax+bx-ab}$.

24. $\dfrac{a^3+b^3}{a^2+ab+ac+bc}$.

25. $\dfrac{x^4-4x^2-3x-6}{x^3+3x^2+x-2}$.

26. $\dfrac{x^4-2x^3-13x^2+22x-8}{x^4-6x^3+12x^2-19x+12}$.

27. $\dfrac{2a^4-13a^3b+31a^2b^2-38ab^3+24b^4}{2a^3-a^2b+ab^2+4b^3}$.

28. $\dfrac{6x^4-7a^2x^2+a^4}{4x^4-9a^2x^2+6a^3x-a^4}$.

29. $\dfrac{6a^3+19a^2b+8ab^2-5b^3}{15a^4+10a^3b+4a^2b^2+6ab^3-3b^4}$.

30. $\dfrac{27a^5b^2-18a^4b^2-9a^3b^2}{36a^6-18a^5-27a^4+9a^3}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

To change a fraction into an equivalent one with a given denominator: multiply the numerator and the denominator by the quotient arising from the division of the given denominator by the denominator of the given fraction.

Thus to change $\frac{2x}{3y}$ into an equivalent fraction with $6y^2z$ for denominator, divide $6y^2z$ by $3y$ and the quotient is $2yz$. Multiply both terms of the given fraction by $2yz$ and we have

$$\frac{2x}{3y}=\frac{4xyz}{6y^2z}.$$

To change two or more fractions into equivalent ones with a common denominator: multiply each numerator by all the denominators except its own.

$$\text{Thus } \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ become } \frac{bc}{abc}, \frac{ac}{abc}, \frac{ab}{abc}.$$

A whole number may be considered a fraction which has 1 for its denominator.

$$\text{Thus } \frac{6}{x+1}, a, \frac{4}{x-1}, \text{ become } \frac{6x-6}{x^2-1}, \frac{ax^2-a}{x^2-1}, \frac{4x+4}{x^2-1}.$$

To change two or more fractions into equivalent ones having their least common denominator: multiply each numerator by the quotient arising from the division of the L.C.M. of the denominators by its denominator.

Thus, to change $\frac{a}{2b}, \frac{b}{4c}, \frac{c}{6d}$ into equivalent fractions having their lowest common denominator,

Find the L.C.M. of the denominators, which is $12bcd$.

Divide $12bcd$ by $2b$ the denominator of the first fraction, and multiply the numerator a by the quotient $6cd$. Thus the first fraction becomes $\frac{6acd}{12bcd}$.

In like manner the multiplier for the numerator of the second fraction is found to be $3bd$, and the fraction becomes $\frac{3b^2d}{12bcd}$.

The multiplier of the third numerator is $2bc$, and the fraction becomes $\frac{2bc^2}{12bcd}$.

To add fractions—Change them, if necessary, into equivalent ones having their least common denominator, and add the numerators.

To subtract one fraction from another—Change them, if necessary, into equivalent ones having their least common denominator, and take the difference of the numerators.

EXAMPLE 1.—Add together $\frac{x}{4}, \frac{x-2}{4}, \frac{x+3}{4}$.

$$\frac{x}{4}+\frac{x-2}{4}+\frac{x+3}{4}=\frac{x+x-2+x+3}{4}=\frac{3x+1}{4}.$$

EXAMPLE 2.—Add together $\frac{a}{a+x}, \frac{a}{a-x}, \frac{ax}{a^2-x^2}$.

$$\frac{a}{a+x}+\frac{a}{a-x}+\frac{ax}{a^2-x^2}=\frac{a^2-ax+a^2+ax+ax}{a^2-x^2}=\frac{2a^2+ax}{a^2-x^2}.$$

Example 3.—Subtract $\dfrac{x^2+1}{x+1}$ from x.

$$x-\frac{x^2+1}{x+1}=\frac{x^2+x-(x^2+1)}{x+1}=\frac{x^2+x-x^2-1}{x+1}=\frac{x-1}{x+1}.$$

Example 4.—Simplify $\dfrac{3+2x}{2-x}-\dfrac{2-3x}{2+x}-\dfrac{16x-x^2}{4-x^2}$.

$$\begin{aligned}&\frac{3+2x}{2-x}-\frac{2-3x}{2+x}-\frac{16x-x^2}{4-x^2}\\
&=\frac{(2+x)(3+2x)-(2-x)(2-3x)-(16x-x^2)}{4-x^2}\\
&=\frac{(6+7x+2x^2)-(4-8x+3x^2)-(16x-x^2)}{4-x^2}\\
&=\frac{6+7x+2x^2-4+8x-3x^2-16x+x^2}{4-x^2}\\
&=\frac{2-x}{4-x^2}=\frac{1}{2+x}.\end{aligned}$$

The signs of the numerator and the denominator of a fraction may be changed without altering the value of the fraction. Thus—

$$\frac{a}{b}=\frac{-a}{-b},\text{ and }\frac{a-b}{b-c}=\frac{b-a}{c-b}.$$

Also

$$\frac{(a-b)\,(b-c)}{(x-y)\,(y-z)}=\frac{(a-b)\,(c-b)}{(x-y)\,(z-y)}=\frac{(b-a)\,(c-b)}{(y-x)\,(z-y)},$$

for if the signs of one factor of a product be changed, all the signs of the product are thereby changed.

Example 5.—Simplify $\dfrac{a}{a^2-x^2}+\dfrac{x}{x^2-a^2}$.

$$\frac{a}{a^2-x^2}+\frac{x}{x^2-a^2}=\frac{a}{a^2-x^2}-\frac{x}{a^2-x^2}=\frac{a-x}{a^2-x^2}=\frac{1}{a+x}.$$

Example 6.—Simplify $\dfrac{a}{(a-b)\,(a-c)}+\dfrac{b}{(b-a)\,(b-c)}+\dfrac{c}{(c-a)\,(c-b)}$.

$$\begin{aligned}&\frac{a}{(a-b)\,(a-c)}+\frac{b}{(b-a)\,(b-c)}+\frac{c}{(c-a)\,(c-b)}\\
&=\frac{a}{(a-b)\,(a-c)}-\frac{b}{(a-b)\,(b-c)}+\frac{c}{(a-c)\,(b-c)}\\
&=\frac{a(b-c)-b(a-c)+c(a-b)}{(a-b)\,(a-c)\,(b-c)}\\
&=\frac{ab-ac-ab+bc+ac-bc}{(a-b)\,(a-c)\,(b-c)}=\frac{0}{(a-b)\,(a-c)\,(b-c)}=0.\end{aligned}$$

Notice that in the third fraction the signs of both factors of the denominator have been changed, and that this does not affect the sign of the numerator.

Exercises XIV.

Change the following fractions into equivalent ones having their lowest common denominator:—

1. $\frac{a}{2}, \frac{b}{3}, \frac{c}{4}$.

2. $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}, \frac{x}{5}$.

3. $\frac{a}{2b}, \frac{b}{4c}, \frac{c}{6d}$.

4. $\frac{1}{x^3y^2z}, \frac{2}{xy^3z^2}, \frac{3}{x^2yz^3}$.

5. $\frac{2a}{3x^2}, \frac{3b}{5y^2z}, \frac{4c}{9xz^2}$.

6. $\frac{1}{a+b}, \frac{1}{a-b}, \frac{a}{a^2-b^2}$.

7. $\frac{a^2+x^2}{a^3-x^3}, \frac{1}{a-x}, \frac{a}{a^2+ax+x^2}$.

8. $\frac{3x}{x^2-4}, \frac{4}{x+2}, \frac{5}{x-2}, \frac{3x}{x^2+4x+4}$.

9. $\frac{x+3}{x^2+6x+8}, \frac{x-1}{x^2+5x+6}, \frac{x+2}{x^2+7x+12}$.

10. $\frac{a}{(a-b)(a-c)}, \frac{b}{(a-b)(b-c)}, \frac{c}{(a-c)(b-c)}$.

Add together—

11. $\frac{4a}{a+1}, \frac{3a}{a-1}, \frac{2a^2}{a^2-1}$.

12. $\frac{3}{x+2}, \frac{5}{x-3}$.

13. $\frac{4x}{2x-3}, \frac{3x}{2x+3}$.

14. $x, \frac{x^2+xy}{x-y}$.

15. $\frac{a}{4}, \frac{3a^2}{a+5}$.

16. $\frac{x}{x+y}, \frac{y}{x-y}, \frac{1}{x}$.

17. $\frac{4x+3y}{x^2+2xy+y^2}, \frac{3x-2y}{x^2-y^2}$.

18. $\frac{1}{x+3}, \frac{2x}{x^2-3x+9}, \frac{3x^2}{x^3+27}$.

19. $\frac{3x}{2x+2a}, \frac{5a}{3x-3a}, \frac{ax}{x^2-a^2}$.

20. $\frac{1}{x^2-5x-14}, \frac{2}{x^2-4}, \frac{3}{x^2-9x+14}$.

21. $x, \frac{x^2}{x+1}, \frac{x^2}{x-1}$.

22. $a+2, \frac{a^2}{a-1}, \frac{a^4}{a^3-1}$.

23. $\frac{x-1}{x+1}, \frac{x-2}{x+2}, \frac{x-3}{x+3}$.

24. $\frac{x^2}{xy+y^2}, \frac{x^2+y^2}{xy}, \frac{y^2}{x^2+xy}$.

25. From $\frac{2a+3b}{7}$ take $\frac{a-2b}{7}$.

26. From $\frac{3x-4}{6}$ take $\frac{x+1}{5}$.

27. " $\frac{1}{x-2}$ " $\frac{1}{x-5}$.

28. " $\frac{x}{x-1}$ " $\frac{x-1}{x}$.

29. " $\frac{3xy}{x^2-y^2}$ " $\frac{y}{x+y}$.

30. " $\frac{a-b}{a^2-ab+b^2}$ " $\frac{a+b}{a^2+ab+b^2}$.

31. " $\frac{1}{x+a}$ " $\frac{x^2+a^2}{x^3+a^3}$.

32. " $\frac{x+2}{x^2+x-30}$ " $\frac{x-3}{x^2-6x+5}$.

Simplify—

33. $x-\frac{2x^2}{x+1}+\frac{x^2}{x+2}$.

34. $\dfrac{x+1}{x+5}-\dfrac{x+5}{x+1}+\dfrac{1}{x^2+6x+5}$.

35. $\dfrac{1}{(x-4)(x-5)}-\dfrac{2}{(x-4)(x-6)}+\dfrac{1}{(x-5)(x-6)}$.

36. $\dfrac{x+3}{x^2-7x+10}+\dfrac{x+2}{x^2-8x+15}+\dfrac{x+5}{x^2-5x+6}$.

37. $\dfrac{1}{x}-\dfrac{2(x-2)}{x^2+3x}+\dfrac{x-3}{x^2+2x}$.

38. $\dfrac{1}{x-1}+\dfrac{x-1}{x^2+x+1}-\dfrac{x^2-x+1}{x^3-1}$.

39. $\dfrac{1}{x-5}-\dfrac{x^2-5x+25}{x^3-125}$.

40. $\dfrac{1}{x+3}-\dfrac{2x}{x^2-9}+\dfrac{x^2}{x^3+27}$.

41. $\dfrac{1}{a+5x}+\dfrac{3(a-x)}{(a+5x)^2}+\dfrac{8a^2-22ax-10x^2}{(a+5x)^3}$.

42. $\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}+\dfrac{2ax}{(a-x)^2}+\dfrac{a^2+x^2}{(a+x)^2}$.

43. $\dfrac{4ab(a^4-b^4)}{a^6-b^6}-\dfrac{a^2-ab+b^2}{a^2+ab+b^2}+\dfrac{a^2+ab+b^2}{a^2-ab+b^2}$.

44. $\dfrac{3}{(a+b)(b+c)}-\dfrac{2}{(a+b)(a+c)}-\dfrac{1}{(a+c)(b+c)}$.

45. $\dfrac{a}{a-b}+\dfrac{3a}{a+b}+\dfrac{2ab}{b^2-a^2}$.

46. $\dfrac{2a^2}{x^2-4a^2}-\dfrac{x}{x+2a}+\dfrac{a}{2a-x}$.

47. $\dfrac{1}{(x-a)(a-b)}-\dfrac{1}{(x-b)(b-a)}$.

48. $\dfrac{a}{(x-a)(a-b)}+\dfrac{b}{(x-b)(b-a)}$.

49. $\dfrac{a^3}{(a-b)(x+b)}+\dfrac{b^3}{(b-a)(x+a)}$.

50. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-a)(b-c)}+\dfrac{1}{(c-a)(c-b)}$.

MULTIPLICATION AND DIVISION OF FRACTIONS.

(A.) To multiply a fraction by a quantity which is not fractional in form, *e.g.* $\dfrac{a}{b}\times c$.

$\dfrac{a}{b}\times c$ means the sum of $\dfrac{a}{b}$ taken c times, that is 'a' of those parts into which unity is divided by b, are to be taken c times.

The sum is therefore ca of those parts, and is expressed thus: $\dfrac{ca}{b}$.

That is $\frac{a}{b} \times c = \frac{ca}{b}$.

RULE.—Multiply the numerator of the fraction by the multiplier.

(B.) To divide a fraction by a quantity not fractional in form, *e.g.* $\frac{a}{b} \div c$.

If $\frac{a}{b}$ is divided by c, the quotient must be such that if it be multiplied by c the product will be $\frac{a}{b}$.

We know from (A) that $\frac{a}{cb} \times c = \frac{ac}{cb}$.

$$\text{That is } \frac{a}{cb} \times c = \frac{a}{b}.$$

$$\left.\begin{array}{r}\text{Dividing each}\\ \text{side by } c\end{array}\right\} \frac{a}{cb} = \frac{a}{b} \div c.$$

RULE.—Multiply the denominator of the fraction by the divisor.

(C.) To multiply a fraction by a fraction, *e.g.* $\frac{a}{b} \times \frac{c}{d}$.

Evidently $c = d$ times $\frac{c}{d}$.

If, therefore, we multiply $\frac{a}{b}$ by c, we multiply by a quantity d times too great. The product must therefore be divided by d to give the correct product of $\frac{a}{b} \times \frac{c}{d}$.

$$\text{First step, } \frac{a}{b} \times c = \frac{ac}{b}.$$

$$\text{Second step, } \frac{ac}{b} \div d = \frac{ac}{bd}.$$

$$\therefore \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

RULE.—In multiplying together two fractions multiply the numerators to form the numerator, and the denominators to form the denominator of the product.

(D.) To divide a fraction by a fraction, *e.g.* $\frac{a}{b} \div \frac{c}{d}$.

Because $c = d$ times $\frac{c}{d}$, therefore, if we divide by c instead of by $\frac{c}{d}$, our divisor is d times too great. The quotient is thus d times too small. This quotient must therefore be multiplied by d to give the correct quotient of $\frac{a}{b} \div \frac{c}{d}$.

First step, $\dfrac{a}{b} \div c = \dfrac{a}{bc}$.

Second step, $\dfrac{a}{bc} \times d = \dfrac{ad}{bc}$.

$$\therefore \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

Rule.—To divide a fraction by a fraction multiply by the divisor inverted.

Exercises XV.

Multiply together—

1. $\dfrac{4x^2+8x}{3x+9}, \dfrac{6x+18}{5x^2+10x}$.
2. $\dfrac{(a-x)^2}{x+y}, \dfrac{x^2-y^2}{a^3-x^3}$.
3. $\dfrac{x^2+5x+6}{4x^2}, \dfrac{6x}{x^2+x-2}$.
4. $\dfrac{x-y}{x+y}, \dfrac{x^2+xy+y^2}{x^2-xy+y^2}$.
5. $\dfrac{x-y}{2}, \dfrac{4}{x^2-y^2}, x+y$.
6. $\dfrac{3x^2-6xy}{5y^2z}, \dfrac{4z^2}{x^2-4y^2}, \dfrac{10y}{9x}$.

Divide—

7. $\dfrac{x^2-y^2}{xy}$ by $\dfrac{x-y}{y}$.
8. $\dfrac{x^2-9}{4}$ " $\dfrac{x+3}{12x}$.
9. $\dfrac{(x+y)^2}{4x-4y}$ " $\dfrac{x-y}{6x+6y}$.
10. $\dfrac{ax^2-x^3}{a+x}$ " $\dfrac{3x}{a^2-x^2}$.
11. $\dfrac{a^2+ab}{a^2b-ab^2}$ " $\dfrac{a^2+2ab+b^2}{a^2b-b^3}$.
12. $\dfrac{x^2-16}{x^2-25}$ " $\dfrac{x+4}{x-5}$.

Multiply together—

13. $\dfrac{x^2-6x+8}{x^2+4x+3}, \dfrac{x^2+8x+15}{x^2-8x+12}, \dfrac{x^2-5x-6}{x^2+x-20}$.
14. $\dfrac{a^2-2ax+x^2}{4a^2x^2}, \dfrac{a^2+x^2}{a^2-ax}, \dfrac{a^3x^3}{a^4-x^4}$.
15. $\dfrac{a^2+2ab+b^2-c^2}{a^2-2ab+b^2-c^2}, \dfrac{a-b+c}{a+b-c}$.
16. $\dfrac{1-a^2+2ab-b^2}{b^2-1-2a-a^2}, \dfrac{1+a+b}{1-a+b}$.
17. $\dfrac{x^2-y^2-z^2+2yz}{x^2-y^2-z^2-2yz}, \dfrac{x^2+y^2-z^2+2xy}{x^2+y^2-z^2-2xy}$.
18. $\dfrac{x^2+ax+bx+ab}{x^2-ax-bx+ab}, \dfrac{x^2-a^2}{x^2-b^2}$.
19. $\dfrac{abx^2+bcdx-aefx-cdef}{x^2-abx+x-ab}, \dfrac{bx^2-ab^2x+efx-abef}{ax^2+cdx-ax-cd}$.
20. Divide $\dfrac{x^2+9xy+20y^2}{x^2-xy-20y^2}$ by $\dfrac{x^3+125y^3}{x^3-4x^2y}$.
21. " $\dfrac{(x+4)^2(x^2-9)}{x^2+2x-15}$ " $\dfrac{x^2+7x+12}{x+5}$.

22. Divide $\dfrac{x^4-20x^2y^2+64y^4}{9xy(x+2y)^2}$ by $\dfrac{x^2-6xy+8y^2}{3x^2}$.

23. " $\dfrac{(x-y)(x^2-xy+y^2)}{x^3+y^3}$ " $\dfrac{x^3-y^3}{(x+y)(x^2+xy+y^2)}$.

24 " $\dfrac{p^2+q^2-r^2+2pq}{r^2-p^2-q^2+2pq}$ " $\dfrac{p+q+r}{q+r-p}$.

25. " $\dfrac{4x^2+y^2-z^2+4xy}{4x^2-y^2-z^2-2yz}$ " $\dfrac{2x+y-z}{3x+y+z}$.

ADDITIONAL SIMPLE EQUATIONS.

The following examples of more difficult simple equations may now be studied with advantage:—

EXAMPLE 1.—Given $\dfrac{a}{x}+\dfrac{b}{x}=a-b$. Find x.

$$\text{Here } \frac{1}{x}(a+b)=a-b.$$

$$\therefore \frac{1}{x}=\frac{a-b}{a+b}.$$

$$\therefore x=\frac{a+b}{a-b}.$$

EXAMPLE 2.—Given $\dfrac{x(a+b)}{a-b}+\dfrac{x(a-b)}{a+b}=\dfrac{a^2+b^2}{a-b}$. Find x.

$$\text{Here}\quad x\left(\frac{a+b}{a-b}+\frac{a-b}{a+b}\right)=\frac{a^2+b^2}{a-b}.$$

$$\text{That is } x\frac{(a+b)^2+(a-b)^2}{a^2-b^2}=\frac{a^2+b^2}{a-b}.$$

$$\text{That is}\quad \frac{2x(a^2+b^2)}{a^2-b^2}=\frac{a^2+b^2}{a-b}.$$

$$\therefore\quad x=\frac{a^2-b^2}{2(a-b)}=\frac{a+b}{2}.$$

EXAMPLE 3. $3a(x-2a)=7b(x-3b)-5ab.$

$$\therefore x(3a-7b)=6a^2-5ab-21b^2.$$

$$\text{That is } x(3a-7b)=(3a-7b)(2a+3b).$$

$$\therefore\quad x=2a+3b.$$

EXAMPLE 4. $\dfrac{x-a}{x+b}=\dfrac{x+c}{x-d}$.

$$\therefore (x-a)(x-d)=(x+b)(x+c).$$

$$\text{That is } x^2-x(a+d)+ad=x^2+x(b+c)+bc$$

$$-x(a+d+b+c)=bc-ad.$$

$$\therefore x=\frac{ad-bc}{a+b+c+d}.$$

EXAMPLE 5. $\frac{x-a}{bc}+\frac{x+b}{ca}-\frac{x-c}{ab}=\frac{2}{a}$.

L.C.M. of denominators is abc.

$\therefore ax-a^2+bx+b^2-cx+c^2=2bc$.

$$\begin{aligned}\therefore x(a+b-c)&=a^2-(b^2-2bc+c^2)\\&=a^2-(b-c)^2\\&=\{a-(b-c)\}\{a+(b-c)\}.\end{aligned}$$

$\therefore \quad x=a-b+c$.

EXAMPLE 6. $\frac{3x^2-4x+5}{3x-4}=\frac{4x^2-5x+7}{4x-5}$.

$$\begin{aligned}\text{That is}\quad x+\frac{5}{3x-4}&=x+\frac{7}{4x-5}.\\ \therefore\quad \frac{5}{3x-4}&=\frac{7}{4x-5}.\\ \therefore\quad 20x-25&=21x-28.\\ \therefore\quad x&=3.\end{aligned}$$

EXAMPLE 7. $\frac{1}{x-2}-\frac{1}{x-5}+\frac{3}{x(x-6)}=0$.

$$\begin{aligned}\text{That is}\quad \frac{1}{x-2}-\frac{1}{x-5}&=-\frac{3}{x(x-6)}.\\ \therefore\quad \frac{(x-5)-(x-2)}{(x-2)\times(x-5)}&=\frac{-3}{x(x-6)}.\\ \text{That is}\quad \frac{-3}{(x-2)(x-5)}&=\frac{-3}{x(x-6)}.\\ \therefore\quad (x-2)(x-5)&=x(x-6).\\ \text{That is}\quad x^2-7x+10&=x^2-6x.\\ \therefore\quad x&=10.\end{aligned}$$

EXAMPLE 8. $\frac{8x-30}{2x-7}+\frac{5x-4}{x-1}=\frac{4x-13}{x-3}+\frac{10x-3}{2x-1}$.

That is $\left(4-\frac{2}{2x-7}\right)+\left(5+\frac{1}{x-1}\right)=\left(4-\frac{1}{x-3}\right)+\left(5+\frac{2}{2x-1}\right)$.

$$\begin{aligned}\therefore\quad \frac{-2}{2x-7}+\frac{1}{x-1}&=-\frac{1}{x-3}+\frac{2}{2x-1}.\\ \therefore\quad \frac{-5}{(2x-7)(x-1)}&=\frac{-5}{(x-3)(2x-1)}.\\ \therefore\quad (2x-7)(x-1)&=(x-3)(2x-1).\\ \text{That is}\quad 2x^2-9x+7&=2x^2-7x+3.\\ \therefore\quad -2x&=-4.\\ x&=2.\end{aligned}$$

The following examples although of a different form are simple equations, because they also can be so simplified that the unknown quantity occurs only in its first power.

EXAMPLE 9. $\sqrt{x}=3$.
Squaring both members $x=9$.

EXAMPLE 10. $\sqrt{x-a^2}=b$.
Squaring both members $x-a^2=b^2$.
$$\therefore x=a^2+b^2.$$

EXAMPLE 11. $\sqrt{10+2x}+\sqrt{3+2x}=7$.
Squaring both members we have—
$$10+2x+3+2x+2\sqrt{10+2x}\sqrt{3+2x}=49.$$
Transposing $2\sqrt{10+2x}\sqrt{3+2x}=36-4x$.
$$\therefore \sqrt{10+2x}\sqrt{3+2x}=18-2x.$$
Squaring $(10+2x)(3+2x)=(18-2x)^2$.
That is $4x^2+26x+30=324-72x+4x^2$.
$$\therefore 98x=294$$
$$\therefore x=3.$$

Examples 9, 10, and 11 are said to be simple equations involving **surds.**

Exercises XVI.

1. $\dfrac{x}{a}=b$.
2. $\dfrac{b}{x}=a$.
3. $\dfrac{ab}{ax+bx}=1$.
4. $\dfrac{a}{x}-\dfrac{b}{x}=(a+b)$.
5. $\dfrac{x}{a}=x-a+\dfrac{1}{a}$.
6. $\dfrac{x}{c}+\dfrac{x}{d}=1$.
7. $a(x-a)+b(2x-3a)=2b^2$.
8. $a^2(x-1)+2a(x-3)+3(x-3)=0$.
9. $2ax+3b(4a-x)=8a^2-5c(x-4a)$.
10. $x^2=(x-1)(x+7)$.
11. $\dfrac{2x^2}{(x-3)}=2x+15$.
12. $\dfrac{x-1}{x+1}=\dfrac{x-6}{x-3}$.
13. $(x-5)(x+6)=(x-12)(x+4)$.
14. $\dfrac{1}{(x+3)(x+5)}=\dfrac{1}{(x+9)(x-5)}$.
15. $\dfrac{a}{b}(a-x)-\dfrac{b}{a}(x+b)=x$.
16. $(a-x)(b-x)=(c+x)(d+x)$.
17. $mn\left(\dfrac{1}{m^2x}-\dfrac{1}{n^2x}\right)=n^2-m^2$.

18. $\frac{1}{x+2}-\frac{2}{2-x}=\frac{3}{1+x}$.

19. $\frac{5x^2-9x+1}{5x-9}=\frac{7x^2-17x-3}{7x-17}$.

20. $\frac{5x+10\frac{1}{2}}{x+\frac{1}{2}}+\frac{2x}{2x+1}=9$.

21. $\sqrt{4x^2+35x+60}=2(x+5)$.

22. $\sqrt{x+4}=\sqrt{x+20}-2$.

23. $\sqrt{9-5x}+\sqrt{6-5x}=3$.

24. $\sqrt{34-x}=\sqrt{18-x}+2$.

25. $\sqrt{x+1}+\sqrt{x-6}=\frac{21}{\sqrt{x-6}}$.

26. $\frac{7x+23}{21}-\frac{x+1}{3}=\frac{1}{2}\left(\frac{x+9}{2x+7}+\frac{23}{35}\right)$.

27. $\frac{2x+3}{5}-\frac{6x+22}{15}=\frac{3x+17}{5(1-x)}$.

28. $\frac{13x-10}{36}+\frac{4x+9}{18}-\frac{7(x-2)}{12}=\frac{13x-28}{17x-66}$.

29. $\frac{11}{12x+17}+\frac{5}{6x+8}-\frac{7}{4x+9}=0$.

30. $\frac{3}{(3x+2)(2x+3\frac{2}{3})}=\frac{5}{(5x-2\frac{1}{3})(2x+5\frac{2}{3})}$.

PROBLEMS.

The following are examples of problems whose solutions are easily obtained by the equational method.

EXAMPLE 1.—*Find three numbers such that the sum of the first and second is* 18, *of the second and third* 22, *of the first and third* 20.

Let $x=$ first number.

$\therefore\ 18-x=$ second number,

$20-x=$ third number.

$\because\ 22=$ sum of second and third,

$\therefore\ 22=(18-x)+(20-x)$.

$\therefore\ 2x=16$,

$x=8$.

The numbers are thus, 8, 10, and 12.

EXAMPLE 2.—*£1000 is lent out part at* $2\frac{1}{2}$, *part at* 3 *per cent. The interest is £26. Find the amount lent at* $2\frac{1}{2}\%$.

Let $x=$ amount at $2\frac{1}{2}\%$.

Then $1000-x=$ amount at 3%,

$x\times\frac{2\frac{1}{2}}{100}=$ interest on the one part.

$(1000-x)\times\frac{3}{100}=$ interest on the other parts.

According to the terms of the problem—

$$\frac{2\frac{1}{2}}{100}x+\frac{3}{100}(1000-x)=26.$$

That is, $2\frac{1}{2}x+3000-3x=2600.$

$$\therefore \frac{x}{2}=400.$$

$$\therefore x=800.$$

The amount lent at $2\frac{1}{2}$°/ₒ is £800.

EXAMPLE 3.—*A number consists of two digits of which the right digit is three times the left digit. If* 18 *be added to the number the digits are reversed. What is the number?*

Let x = the left digit,

$3x$ = the right digit.

Just as $2\times 10+3$ is the number whose digits are 2 and 3, so $(10x+3x)$ is the number required.

And $(30x+x)$ is the number we obtain by reversing the digits.

According to the terms of the problem, we have—

$$10x+3x+18=30x+x,$$

$$13x+18=31x.$$

$$\therefore x=1,$$

and $3x=3$.

The number is thus 13.

EXAMPLE 4.—*At the same time that the up-train going at the rate of* 33 *miles an hour passes A, the down-train going at the rate of* 21 *miles an hour passes B. They collide* 18 *miles beyond the midway station. How far is A distant from B?*

Let x = the distance in miles.

$\frac{x}{2}$ = half the distance.

$\frac{x}{2}+18$ = number of miles the up-train goes.

$\frac{x}{2}-18$ = number of miles the down-train goes.

Now $\frac{\text{distance in miles}}{\text{rate in miles per hour}}$ = number of hours of travel.

$\frac{\frac{x}{2}+18}{33}$ = time the up-train takes.

$\frac{\frac{x}{2}-18}{21}$ = time the down-train takes.

$$\therefore \frac{\frac{x}{2}+18}{33}=\frac{\frac{x}{2}-18}{21}.$$

Solving $x=162$.

The distance between A and B is 162 miles.

Exercises XVII.

1. The sum of two numbers is 48, and their difference is 6. What are the numbers?
2. The difference of two numbers is 4, and their product exceeds the square of the less by 48. Find the numbers.
3. One number exceeds another by 3, and its square exceeds that of the other by 33. What are the numbers?
4. The difference of the squares of two consecutive numbers is 25. Find the numbers.
5. There are in a school 144 pupils. If there were twice as many boys and half as many girls, there would be 156 pupils. What was the number of boys and girls respectively?
6. A has 10*s*. more than B, and C has one-fourth more than A and B together. Their joint-purse amounts to £36. How much has C?
7. A train has 60 passengers; the first-class passengers pay 2*d*. per mile, the third-class 1*d*. per mile; the distance travelled is 8 miles, and the total receipt is £3. How many passengers are there of each class?
8. A's age is thrice that of B; and B's age is as much below 30 as A's is above it. Find B's age.
9. A cask contains four times as much wine as water, but if 8 gallons of wine and 6 gallons of water be added, there will be only three times as much wine as water. How much water is there in the cask?
10. The ages of a father and son together amount to 70; and two years ago the father was five times as old as the son. What is the age of each?
11. One-fourth of a class are at writing, one-third are at arithmetic, and a pupil-teacher attends to the remaining 25. How many are there in the class?
12. A man bequeaths $\frac{1}{2}$ of his property to his son, $\frac{1}{3}$ of the remainder to each of two daughters, and the remaining £2000 to his widow. What was the property worth?
13. In travelling 24 miles by steamer and 12 by rail, I take 15 minutes more than if I had gone 18 miles by steamer and 18 by rail. If the steamer goes at half the rate of the train, find the rate of each.
14. The united ages of a father, mother, son, and daughter amount to 130 years. The father is 10 years older than the mother. The latter is 26 years older than the daughter, who is 4 years younger than her brother. Find the father's age.
15. A church collection of £10 is made up of half-crowns, sixpences, and pence. There are 3 times as many half-crowns as sixpences, and 6 times as many pennies as half-crowns and sixpences together. What was the amount of the pence?

16. A sum of money is stolen from two tills containing 8*s.* 4*d.* and 4*s.* respectively. From the first twice as much is taken as from the second, and the sum remaining in the first is thrice that in the second. How much was taken from each?
17. A man walks a certain distance in 9 hours. If his rate had been half a mile more per hour his time would have been one hour less. How far did he walk?
18. How much water must be mixed with 20 gallons of whisky at 15*s.* a gallon to make mixture worth 12*s.* a gallon?
19. The perimeter of a rectangular field is 280 yards. If its length were 2 yards more and its breadth 2 yards less, its area would be 44 yards less. What are its dimensions?
20. A man's income of £500 is made up partly from salary and partly from profits. If his salary were increased one-fifth, and his share in the profits doubled, he would get £180 more. What salary did he receive?

SIMULTANEOUS EQUATIONS.

If we express algebraically that the sum of two numbers, each of which is unknown, is equal to 30, we have the equation—

$$x+y=30.$$

Here, one unknown quantity is represented by x, and the other unknown quantity is represented by y. The equation—

$$x+y=30,$$

is said to be an equation between two unknown quantities. From such an equation we can determine nothing definite regarding x and y, except that whatever x may be, $y=30-x$.

For example, if $x=2$, $y=30-2=28$;
if $x=20$, $y=30-20=10$;
if $x=1000$, $y=30-1000=-970$.

Such an equation is termed an **indeterminate equation.**

But if at the same time that—

$$x+y=30,$$

another relation exists between the unknown quantities, such as—

$$2x+y=50,$$

then in determining x and y we have to find not only values which satisfy the equation—

$$x+y=30,$$

but which at the same time satisfy the second condition—

$$2x+y=50.$$

In such a case as this the values of x and y are not indeterminate but fixed.

For since, according to the first equation—

$$y = 30 - x,$$

we may substitute this value of y in the second equation; and there results the simple equation—

$$2x + (30 - x) = 50.$$

Solving this equation, we get—

$$2x - x = 50 - 30,$$
$$\text{or } x = 20;$$
$$\text{and since } y = 30 - x$$
$$\therefore\ y = 30 - 20 = 10.$$

If 20 be substituted for x, and 10 be substituted for y in the two equations, we get the two numerical identities—

$$(1.)\ 20 + 10 = 30.$$
$$(2.)\ 2 \times 20 + 10 = 50.$$

The substitution of no other values for x and y in the two equations would **simultaneously** produce numerical identities.

In the above we have shown how to solve simultaneous equations by the method of substitution. We expressed from one equation the value of y in terms of x, and substituting this value of y in the other equation we found a new equation in which y was not involved. This simple equation gave us the value of x, and y was determined by substitution.

Every method of solving simultaneous equations involving two unknown quantities aims at the formation of a new equation in which only one unknown quantity is involved.

EXAMPLE 1. $\quad 9x - 2y = 51 \ldots\ldots\ldots(1)$

$\qquad\qquad\quad 11x + 8y = 31 \ldots\ldots\ (2)$

Multiplying equation (1) by 4, we have—

$$36x - 8y = 204.$$
$$\text{Now, } 11x + 8y = 31.$$

Adding the two left-hand and the two right-hand members of these two equations we have a new equation—

$$47x = 235.$$
$$\therefore\ x = 5.$$

And y is found by substituting 5 for x in either of the original equations.

EXAMPLE 2. $\quad 14x + 15y = 43 \ldots\ldots\ldots(1)$

$\qquad\qquad\quad 9x + 4y = 22 \ldots\ldots\ldots(2)$

Multiplying equation (1) by 4, we have—

$$56x + 60y = 172 \ldots\ldots\ldots(3)$$

Multiplying equation (2) by 15,

$$135x + 60y = 330 \ldots\ldots\ldots(4)$$

And subtracting equation (3) from equation (4), we have—

$$79x = 158,$$
$$\therefore\ x = 2;$$

and y is found by substitution.

From the above we get the following rules:

(1.) Multiply the equations by such quantities as will make the coefficients of one unknown the same in the resulting equations.

(2.) Subtract or add, according as the coefficients are of like or unlike sign.

An equation involving only the other unknown quantity results.

(3.) Solve this equation, and then find the other unknown by substitution.

Another method will be understood from the following example:—

$$6x - 9y = 21 \quad\ldots\ldots\ldots(1)$$
$$3x + 2y = 43 \quad\ldots\ldots\ldots(2)$$

From equation (1) express y in terms of x.

$$\because\ 6x - 9y = 21,$$
$$\therefore\ 9y = 6x - 21.$$
$$\therefore\ y = \frac{6x-21}{9} = \frac{2x-7}{3}.$$

From equation (2) express y in terms of x.

$$\because\ 3x + 2y = 43,$$
$$\therefore\ 2y = 43 - 3x.$$
$$\therefore\ y = \frac{43-3x}{2}.$$

Equate these two values of y obtained from equations (1) and (2) respectively.

$$\frac{2x-7}{3} = \frac{43-3x}{2}.$$

Solve—

$$2(2x - 7) = 3(43 - 3x).$$
$$4x - 14 = 129 - 9x.$$
$$4x + 9x = 129 + 14.$$
$$13x = 143.$$
$$\therefore\ x = 11.$$

y is found by substitution.

Exercises XVIII.

1. $\begin{cases} x + y = 11. \\ x - y = 3. \end{cases}$
2. $\begin{cases} 3x + 7y = 47. \\ 5x - 6y = -10. \end{cases}$
3. $\begin{cases} 3x - 2y = 8. \\ 4x + 4y = 44. \end{cases}$
4. $\begin{cases} 2x + 3y = 26. \\ 4x - y = 24. \end{cases}$
5. $\begin{cases} 4x - 3y = 5. \\ 2x - y = 7. \end{cases}$
6. $\begin{cases} 9x + 11y = 129. \\ 18x - 21y = 0. \end{cases}$
7. $\begin{cases} 9x - y = -1. \\ 21x + 2y = 41. \end{cases}$
8. $\begin{cases} 27x - 28y = 25. \\ 17x - 25y = 1. \end{cases}$
9. $\begin{cases} 3x - 2y = 19. \\ x + 5y = 80. \end{cases}$
10. $\begin{cases} 4x - 5y = -71. \\ x + 2y = 343. \end{cases}$
11. $\begin{cases} 5x - 4y = 122. \\ 2x - 2y = 10. \end{cases}$
12. $\begin{cases} 6x + 9y = 21. \\ 5x - 4y = 2\frac{1}{8}. \end{cases}$
13. $\begin{cases} 4x + 7y = 2. \\ 9x + 5y = 2\frac{27}{28}. \end{cases}$
14. $\begin{cases} 2x - 3y = 0. \\ 3x - 5y = -\frac{1}{8}. \end{cases}$

15. $\begin{cases} 5x - 3y = \frac{2}{3}. \\ 6x + 4y = 1\frac{29}{45}. \end{cases}$

16. $\begin{cases} 24x - 55y = -14. \\ 63x + 20y = 29. \end{cases}$

17. $\begin{cases} 9x - 7y = -7. \\ 15x + 2y = 2. \end{cases}$

18. $\begin{cases} 3x - 15y = 4\frac{1}{2}. \\ 7x + 8y = 32. \end{cases}$

19. $\begin{cases} 5\frac{1}{2}x + 1\frac{1}{3}y = \frac{11}{18}. \\ 7\frac{1}{3}x - \frac{3}{2}y = \frac{13}{24}. \end{cases}$

20. $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = 17. \\ \dfrac{y}{4} - \dfrac{x}{6} = 3. \end{cases}$

21. $\dfrac{x}{2} + \dfrac{y}{3} = 5,\ x = y - 5.$

22. $\dfrac{x}{2} - \dfrac{y}{4} = 3\frac{3}{4},\ \dfrac{x}{3} + y = 10\frac{2}{3}.$

23. $\dfrac{x - 2y}{5} = y - 1,\ \dfrac{3x - 4y}{6} = x - 10.$

24. $x - \dfrac{2x - 3y}{3} = 7\frac{2}{3},\ y + \dfrac{5x - 4y}{8} = 7\frac{1}{2}.$

25. $\dfrac{x}{5} - \dfrac{y}{7} = 1,\ \dfrac{y}{2} - \dfrac{x}{3} = 2.$

26. $\dfrac{x + y}{3} - \dfrac{x - y}{5} = 10,\ \dfrac{2x}{3} = \dfrac{3y}{2}.$

27. $\dfrac{5x + 3y}{4} - \dfrac{2y - 1}{3} = x,\ 20 + x - y = 0.$

28. $\dfrac{1 - 2x}{3} + \dfrac{3y - 1}{2} = 3,\ \dfrac{x - 2y}{4} + x = 1.$

29. $y + x - 1 = \dfrac{x + 10y - 5}{6},\ y - x - 1 = 6x - 7y.$

30. $\dfrac{x}{2} - 12 = \dfrac{y}{4} + 8,\ \dfrac{x + y}{5} + \dfrac{x}{3} = \dfrac{1}{4}(2y - 4) + 21.$

31. $x + y + z = 7,\ 2x + 3y + z = 11,\ 3x + y + 2z = 15.$

32. $x + 2y + 3z = 10,\ 2x + 3y + 4z = 16,\ 3x + y + 2z = 13.$

33. $x + 6y + 2z = 29,\ 4x + 3y + z = 18,\ 3x + y + 5z = 31.$

34. $2x - y + 3z = 23,\ 3x + 4y - z = 7,\ 5x - 2y - 2z = 1.$

35. $5x - 2y + 3z = 24,\ 7x + 3y - 2z = 13,\ 2x - y + z = 7.$

36. $3x + 2y - z = 16,\ 5x - y - 2z = 3,\ 2x - 4y + 5z = 18.$

Problems producing Simultaneous Simple Equations.

In the solution of problems where two different relations between two unknown quantities are stated, we express these relations algebraically, and solve the two simultaneous equations thus formed.

Example 1.—*Two numbers are such that if five times the greater be added to the less the sum is* 140; *and if three times the less be subtracted from twice the greater the remainder is* 5. *What are the numbers?*

Let x = the greater number.
y = the less number.

Expressing the first relation algebraically we have—

$$5x + y = 140 \ldots\ldots\ldots(1)$$

Expressing in the same manner the second relation we have—

$$2x-3y=5 \ldots\ldots\ldots\ldots(2)$$

The numbers required are the values of x and y respectively which satisfy the two simultaneous equations—

$$5x+\ y=140 \ldots\ldots\ldots(1)$$
$$2x-3y=\quad 5 \ldots\ldots\ldots(2)$$

To solve, multiply (1) by 3.

$$15x+3y=420.$$

But by equation (2), $\quad 2x-3y=\quad 5.$

Adding, $\quad 17x \quad =425.$

$$\therefore x \quad = \quad 25.$$

Substituting we find $y=15$.

Thus the numbers are 25 and 15.

EXAMPLE 2.—*A steamer goes* 48 *miles up a river in* 6 *hours, and back again in* 3 *hours. What would be the rate of steamer in still water?*

Let x = rate of steamer in still water (miles per hour).
y = rate of current (miles per hour).
$(x+y)$ = rate of steamer with current.
$(x-y)$ = rate of steamer against current.

According to the data we have—

$$6(x-y)=48 \ldots\ldots\ldots(1)$$
$$3(x+y)=48 \ldots\ldots\ldots(2)$$

Solving, $x=12. \quad y=4.$

Rate of steamer in still water is 12 miles per hour. Rate of current is 4 miles per hour.

EXAMPLE 3.—*If* 1 *be added to the numerator and denominator of a certain fraction its value becomes* $\frac{1}{2}$, *and if* 5 *be subtracted from its numerator and denominator its value becomes* $\frac{1}{4}$. *Find the fraction.*

Let x = numerator.
y = denominator.

$$\therefore \frac{x}{y} = \text{fraction.}$$

According to terms of problem we have—

$$\frac{x+1}{y+1}=\frac{1}{2} \ldots\ldots\ldots(1)$$
$$\frac{x-5}{y-5}=\frac{1}{4} \ldots\ldots\ldots(2)$$

From (1) $2x+\ 2=\ y+1.$

$$\therefore \quad y=2x+1.$$

From (2) $4x-20=\ y-5.$

$$\therefore \quad y=4x-15.$$

Thus $4x-15=2x+1.$

$$\therefore \quad 2x=16.$$
$$\therefore \quad x=8.$$

And $\quad y=2x+1=17.$

Therefore the fraction is $\frac{8}{17}$.

EXAMPLE 4.—*A number consists of two digits. If the left digit be increased by 5, and the right digit be diminished by 5, the result will be double the original number. If these new digits be transposed the result will be $\frac{1}{6}$th the original number. Find the number.*

Let x = left digit.
y = right digit.

Now, just as $20+3$ is the number whose left digit is 2, and right digit is 3,

so $10x+y$ = number whose digits are x and y.
$x+5$ = left digit increased by 5.
$y-5$ = right digit diminished by 5.
$\therefore\ 10(x+5)+(y-5)$ = number with these new digits.
$10(y-5)+(x+5)$ = number with these digits transposed.

According to the terms of the problem, we have—

$$10(x+5)+(y-5)=2(10x+y)\ldots\ldots\ldots(1).$$
$$10(y-5)+(x+5)=\tfrac{1}{6}(10x+y)\ldots\ldots\ldots(2).$$

Solving, we get $y=5$.
$x=4$.

Therefore the number is $40+5$, *i.e.* 45.

Exercises XIX.

1. Find two numbers whose sum shall be 130, and difference 44.
2. Find two numbers such that twice the first shall be less than three times the second by 3; and four times the first, greater than five times the second by 5.
3. Find two numbers such that the sum of one-half the first and one-third the second is equal to 14; and one-third of the first diminished by one-fifth of the second is equal to 3.
4. One-third the difference of two numbers is equal to five; and four-fifths of their sum is equal to the greater. What are the numbers?
5. The difference of two numbers is as much below 10 as four times their sum is above 80; and three times the greater is as much above twice the less as four times the less is below 56. What are the numbers?
6. Find a proper fraction such that the difference of its numerator and denominator is equal to one-third of the denominator; and if the numerator be doubled while the denominator is increased by 1, the result is unity.
7. The price of 5 turkeys and 8 geese is £2, 9*s*., while 4 turkeys and 3 geese cost £1, 9*s*. Find the price of a turkey, and of a goose.
8. A man rows 20 miles up a river in 3 hours 20 minutes, and back in 2 hours. Supposing him to row uniformly, calculate the velocity of the current.

SQUARE ROOT.

The **Square Root** of a quantity is that quantity whose square is the given quantity.

Thus, the square root of $9a^4b^2$ is $3a^2b$, because $(3a^2b)^2=9a^4b^2$.

But, since $(-3a^2b)^2$ is also equal to $9a^4b^2$, it follows that the square root of $9a^4b^2$ is either $+3a^2b$ or $-3a^2b$.

This is usually written $\sqrt{9a^4b^2}=\pm 3a^2b$.

Here only the positive values of the roots need be considered.

To find the square root of the product of several factors: take the square root of each factor separately and multiply the results. If any of the factors of the given expression are not complete squares, write these with the root symbol over them.

It follows from the definition that a negative quantity has no square root.

EXAMPLES.

1. $\sqrt{64a^6b^4x^8y^2}=8a^3b^2x^4y.$
2. $\sqrt{a^2b}=a\sqrt{b}.$

The method of extracting the square root of a compound expression will be understood from a consideration of the following examples.

EXAMPLES.

1. Find the square root of $a^2+2ab+b^2$.

$$
\begin{array}{ll}
 & a^2+2ab+b^2(a+b. \quad \text{Ans.} \\
 & a^2 \\
2a+b & \quad 2ab+b^2 \\
 & \quad 2ab+b^2
\end{array}
$$

The square root of the first term of the given expression is a, and this is the first term of the answer. Subtract the square of a from the given expression and the remainder is $2ab+b^2$.

To the left of this put the double of the part of the root already found, that is, $2a$.

Divide the first term of the remainder by $2a$ and the quotient is $+b$, which is the second term of the answer.

Add b to $2a$; multiply the whole by b, and subtract the product from $2ab+b^2$

$a+b$ is the root required.

2. Find the square root of $x^4-8x^3y+22x^2y^2-24xy^3+9y^4$.

$$
\begin{array}{ll}
 & x^4-8x^3y+22x^2y^2-24xy^3+9y^4(x^2-4xy+3y^2. \quad \text{Ans.} \\
 & x^4 \\
2x^2-4xy & \quad -8x^3y+22x^2y^2 \\
 & \quad -8x^3y+16x^2y^2 \\
2x^2-8xy+3y^2 & \qquad\qquad 6x^2y^2-24xy^3+9y^4 \\
 & \qquad\qquad 6x^2y^2-24xy^3+9y^4
\end{array}
$$

The third term of the root is found by placing on the left the double of the first two terms and proceeding as before.

In an example such as the above, in which the terms are or can be arranged according to ascending or descending powers of some common letter, the method of detached coefficients may be employed.

3. Find the square root of $4x^6 - 8x^5 + 20x^3 - 15x^2 - 8x + 16$.

$$
\begin{array}{l|l}
 & 4-8+0+20-15-8+16\,(2-2-1+4 \\
 & 4 \\
\hline
4-2 & -8+0 \\
 & -8+4 \\
\hline
4-4-1 & -4+20-15 \\
 & -4+\ 4+\ 1 \\
\hline
4-4-2+4 & 16-16-8+16 \\
 & 16-16-8+16 \\
\hline
\end{array}
$$

Therefore $2x^3 - 2x^2 - x + 4$ is the square root required.

4. Find the square root of $\frac{x^4}{9} + \frac{2x^3}{3} + \frac{4x^2}{3} + x + \frac{1}{4}$.

$$
\begin{array}{l|l}
 & \frac{x^4}{9} + \frac{2x^3}{3} + \frac{4x^2}{3} + x + \frac{1}{4}\left(\frac{x^2}{3} + x + \frac{1}{2}.\ \text{Ans.}\right. \\
 & \frac{x^4}{9} \\
\hline
\frac{2x^2}{3} + x & \frac{2x^3}{3} + \frac{4x^2}{3} \\
 & \frac{2x^3}{3} + x^2 \\
\hline
\frac{2x^2}{3} + 2x + \frac{1}{2} & \frac{x^2}{3} + x + \frac{1}{4} \\
 & \frac{x^2}{3} + x + \frac{1}{4} \\
\hline
\end{array}
$$

Exercises XX.

Find the square root of—

1. $x^2 + 2xy + y^2$.
2. $4 + 4b + b^2$.
3. $9a^2 + 6a + 1$.
4. $25b^2 + 120b + 144$.
5. $9x^2 + 24xy + 16y^2$.

Find the square root of—

6. $p^2 - 2pq + q^2$.
7. $x^2 - 8xy + 16y^2$.
8. $64x^2 - 144xy + 81y^2$.
9. $49 - 84c + 36c^2$.
10. $x^4 + 2x^3 + 3x^2 + 2x + 1$.
11. $x^4 + 6x^3y + 17x^2y^2 + 24xy^3 + 16y^4$.
12. $9x^4 + 30x^3 + 61x^2 + 60x + 36$.
13. $x^4 - 4x^3 + 6x^2 - 4x + 1$.
14. $x^4 + 8x^3y + 6x^2y^2 - 40xy^3 + 2$
15. $9 - 6x + 18x^2 - 4x^3 + 4x^4$.

16. $x^2 - 2xy + 2xz + y^2 - 2yz + z^2$.
17. $16x^2 - 24xy - 16xz + 9y^2 + 12yz + 4z^2$.
18 $9a^2x^2 - 12abxy + 6acxz + 4b^2y^2 - 4bcyz + c^2z^2$.
19. $x^6 + 4x^5 + 6x^4 + 8x^3 + 9x^2 + 4x + 4$.
20. $4x^6 + 12x^5y + 25x^4y^2 + 28x^3y^3 - 22x^2y^4 + 8xy^5 + 1$.
21. $x^6 - 6x^5y + x^4y^2 + 28x^3y^3 + 4x^2y^4 - 16xy^5 + 4y^6$.
22. $9x^6 - 12x^5 - 2x^4 - 20x^3 + 17x^2 + 8x + 16$.
23. $x^2 + \dfrac{2ax}{3} + \dfrac{a^2}{9}$.
24. $\dfrac{x^2}{4} - xy + y^2$.
25. $\dfrac{a^2}{9} - \dfrac{ab}{6} + \dfrac{b^2}{16}$.
26. $\dfrac{x^4}{16} + \dfrac{x^3}{2} + \dfrac{7x^2}{6} + \dfrac{2x}{3} + \dfrac{1}{9}$.
27. $\dfrac{1}{4} - 2x + \dfrac{13}{4}x^2 + 3x^3 + \dfrac{9}{16}x^4$.
28. $x^4 + x^3 + \dfrac{5}{4}x^2 + \dfrac{5}{2}x + \dfrac{5}{4} + \dfrac{1}{x} + \dfrac{1}{x^2}$.

MORE DIFFICULT FACTORS.

(i) Multiply $2x + 3y$ by $4x + 5y$, and observe that the answer is $8x^2 + 22xy$, *i.e.* $(12 + 10)xy + 15y^2$; multiply $2x - 3y$ by $4x - 5y$, and observe that the answer is $8x^2 - 22xy$, *i.e.* $(12 + 10)xy + 15y^2$.

Notice that this answer differs from the previous answer only in the sign of the second term, that *when both are + the sign of the second term is +, and when both are − the sign of the second term is minus.*

(ii) Now multiply $2x - 3y$ by $4x + 5y$, and notice that the product is $8x^2 - 2xy$, *i.e.* $(10 - 12)xy - 15y^2$.

Observe that in our previous cases we had $+15y^2$, so that *the + before the third term shows that the signs are of the same kind, the minus that they are different.*

(iii) Next multiply $2x + 3y$ by $4x - 5y$, and notice that the answer is $8x^2 + 2xy$, *i.e.* $(12 - 10)xy - 15y^2$. Here the only difference from the last answer is in the sign of the second term, which is + instead of minus; that is, *the sign of the second term shows whether the plus or minus must be greater.*

By applying these principles it is possible to factorize more difficult quantities than those given on pp. 28 and 29.

EXAMPLE: $12x^2 + 19x + 4$. Here the + before the third term shows that the signs are like, the + before the second that they are both plus.

The usual way of solving is by trial, thus $\begin{smallmatrix}6 & & 2\\ & \times & \\ 2 & & 2\end{smallmatrix}$, $12 = 6 \times 2$ or 4×3, and $4 = 2 \times 2$ or 4×1. Try 6, 2 and 2, 2 as shown. These

figures give the right numbers for second and third term, but give $12+4$ (multiplying across) for second. Try $\begin{smallmatrix}6 & & 4\\ & \times & \\ 2 & & 1\end{smallmatrix}$, as before they give for second term only 14, instead of 19. Try $\begin{smallmatrix}4 & & 4\\ & \times & \\ 3 & & 1\end{smallmatrix}$, only 16 for second term. Try $\begin{smallmatrix}3 & & 4\\ & \times & \\ 4 & & 1\end{smallmatrix}$, this gives 19 for second term, so our terms are $(3x+4)\ (4x+1)$. Much time may be saved by adopting the following plan:—

Take the example given above, $12x^2+19x+4$.

$48=$ product of coefficient of first and third terms.

$48=$ also product of two numbers which added together $=19$, *i.e.* of 3 and 16.

$2\times 24=48$, and $3\times 16=48$, but while $3+16=19$, $2+24=26$.

$\therefore$ 3 and 16 are the terms to try.

Proceeding thus, and cancelling, we have—

$$\therefore \frac{(\overset{3}{\cancel{12}}x+\overset{4}{\cancel{16}})\ (\overset{4}{\cancel{12}}x+\overset{1}{\cancel{3}})}{\cancel{12}\ \cancel{3}}=\text{terms}=(3x+4)\ (4x+1).$$

Exercise XXI

Give the factors of—

1. $24x^2-19x+2$.
2. $8x^2-49x+6$.
3. $18x^2+43x-5$.
4. $6x^2-43x-15$.
5. $9x^2-89x-10$.
6. $20x^2-19x+3$.
7. $20x^2+11x-3$.
8. $10x^2-23x+6$.
9. $14x^2-53x+14$.
10. $14x^2-96x-14$.
11. $49x^2+21x-4$.
12. $15x^2-23x+8$.
13. $15x^2-7x-8$.
14. $5x^2-43x+24$.
15. $13x^2-25x+12$.
16. $13x^2-20x-12$.

Exercise XXII

Give the factors of—

1. $11x^2-13xy-24y^2$.
2. $22x^2+35xy+12y^2$.
3. $33x^2-10xy-8y^2$.
4. $9x^2+25xy+16x^2$.
5. $3x^2-7xy-48y^2$.
6. $18x^2+45xy-8y^2$.
7. $10x^2+29xy+10y^2$.
8. $4x^2-21xy-25y^2$.
9. $25x^2-29xy+4y^2$.
10. $8x^2-35xy+12y^2$.
11. $32x^2-4xy-3y^2$.
12. $4x^2-29xy-24y^2$.
13. $16x^2-xy-15y^2$.
14. $8x^2+31xy+30y^2$.
15. $24x^2-83xy+10y^2$.
16. $6x^2-25xy-9y^2$.
17. $2x^2-29xy+27y^2$.
18. $3x^2+55xy+18y^2$.

TEST PAPERS.—FIRST YEAR.

A[1].

1. If $a=5$, $b=4$, $c=3$, $d=2$, find the numerical value of—

$$\frac{2a+b}{b-d}-\frac{3b+c}{c+d}+\frac{4c-d}{a}.$$

2. Add together $3a-2b+c$, $a+3b+2c$, $-5a+b-3c$, $2a-2b-c$.
3. From x^3-5x^2+3x-7 subtract $-x^3+x^2+5x-4$.
4. Multiply $a^2b^2-3abc+5c^2$ by $2ab-c$.
5. Divide a^4-16b^4 by $a+2b$.
6. Solve $\frac{4-x}{2}+\frac{6-2x}{5}=x-12$.

B[1].

1. Find the value of $(a+b)^2+(a-b-c+d)^2+(c+d)^2$, when $a=1$, $b=2$, $c=4$, $d=8$.
2. From $4a^2-3ab+b^2$ take $a^2+ab+2b^2$.
3. Simplify $8a-\{3a+(4a-3)\}$.
4. Find the product of $x^2-2xy+y^2$ and x^2+xy+y^2.
5. Divide $10x^4-9x^3y-13x^2y^2+6xy^3$ by $5x^2-2xy$.
6. Solve $6-\frac{x+1}{4}=5$.

C[1].

1. What is the numerical difference between $a-(b-c)$ and $a-b-c$, when $a=7$, $b=3$, $c=2$?
2. Find the sum of $3m^2-2mn+5n^2$, m^2-n^2, $4mn+2n^2$, $2m^2-2mn$.
3. Simplify $x+1+\{4x-(3x+5)\}$.
4. Multiply together $x+1$, $x-2$, $x+3$.
5. Divide $x^4-p^2x^2+2pq^2x-q^4$ by x^2+px-q^2.
6. Solve $x-\frac{3x+1}{7}=1$.

D[1].

1. Find value of $\frac{x^4+a^2x^2+a^4}{x^2-ax+a^2}$, when $x=0$, $a=1$.
2. Take the sum of $4x^2+3xy-y^2$ and $x^2+2xy+5y^2$ from $8x^2+9y^2$.
3. Simplify $3x^2-\{2x-(4x^2+3x)-1\}$.
4. Multiply $6x^2-7xy+2y^2$ by $x^2+2xy+3y^2$.
5. Divide x^6-y^6 by x^2-xy+y^2.
6. Solve $\frac{x+1}{2}+\frac{x+2}{3}=2$.

E¹.

1. What is the value of $\dfrac{6c^4-96}{2c^3+4c^2+8c+16}$, when $c=3$?
2. Add together $x-2\sqrt{xy}+y$, $x-y$, $x+2\sqrt{xy}+y$.
3. Subtract $3a^2-a+7$ from $5a^3-4a+9$.
4. Simplify $1-(3x-2)-\{x+(5x-4)\}$.
5. Multiply $x-2y+3z$ by $x+2y-3z$, and prove your result by division.
6. Solve $3x-\dfrac{3x-2}{5}=10$.

F¹.

1. Find the value of $ab-6b^2c-99abcd$, when $a=8$, $b=4$, $c=2$, $d=0$.
2. From the sum of $x-y+z$ and $2x-2y-2z$ take $3x-3z$.
3. Simplify $6a-[5a-\{4a-(3a-2)\}]$.
4. Find the continued product of $2a-b$, $2a+b$, $4a^2+b^2$
5. Divide x^4-y^4 by x^2+y^2.
6. Solve $\dfrac{x}{2}-\dfrac{4-x}{3}=3\frac{2}{3}$.

G¹.

1. Distinguish between "power" and "index."
2. Find the value of $\dfrac{x-y}{x^2-xy+y^2}+\dfrac{1}{x+y}-\dfrac{x^2-xy}{x^3+y^3}$, when $x=3$, $y=2$.
3. Add together $2+5a-6b+c$, $a+3b-4c+1$, $b+2c-4-3a$, and subtract the result from $c+3-a+b$.
4. Multiply x^2+xy+y^2 by x^2-xy+y^2.
5. Divide a^5-5a^3+4a by a^2-a-2.
6. Solve $\dfrac{2x-1}{3}+\dfrac{3x+1}{8}=x$.

H¹.

1. Find the value of $\dfrac{x+\dfrac{x^2}{3}+3}{x+\dfrac{9}{x}+3}$, when $x=12$.
2. Take $a-b+c$ from the sum of $a+b+c$, $b+c-a$, $c-a-b$, $a-b-c$.
3. Simplify $9-[3a+\{4b-(6+\overline{a-b})\}]$.
4. Multiply $a^4+2a^3+3a^2+2a+1$ by a^2-2a+1.
5. Divide $x^2-y^2-z^2+2yz+x+y-z$ by $x+y-z$.
6. Solve $2(3x+7)+17=9(x+5)-2$.

I¹.

1. If $x=5$, $y=3$, what is the value of—
$$\frac{2}{x+y}+\frac{2}{x-y}-\frac{4x}{x^2+y^2}+\frac{4y}{x^2-y^2}?$$
2. Simplify $1-[a-\{1-a-\overline{(1-a)}\}]$.

3. Multiply out $2x(4x^2+1)(2x+1)(2x-1)$.
4. Divide $x^3+y^3+z^3-3xyz$ by $x+y+z$.
5. Take $y+z-x$ from $7x+3z+y$.
6. Solve $8-3(x-7)=2x-11$.

J[1].

1. Find the value of $\frac{(15x^2-7x-2)(6x^2+7x+2)}{15x^2+13x+2}$, when $x=2$.
2. Add together $2x-3by$, $ax+4y$, $3x+by$, $y-2ax$.
3. From $ax-y+cz$ take $x-by+3z$.
4. Multiply $1+2x+3x^2+4x^3+5x^4$ by $1-2x+x^2$.
5. Divide x^5+y^5 by $x+y$.
6. Solve $(x+3)(x+4)=(x+1)(x+2)+2$.

K[1].

1. From $2ab-7bc+2cd-5de$ take $ab-3bc+4cd-6de$.
2. If $a=8$, $b=3$, what is the value of $3a-[2b+1-\{a+(4-3b)\}]$?
3. Multiply $x^2-xy+2y^2$ by itself.
4. Divide $x^4+x^2y^2+y^4$ by x^2-xy+y^2.
5. Simplify $x+y-z+\{x-(y-z)\}$.
6. Solve $\frac{3x}{7}-\frac{x-10}{2}=\frac{x}{2}-\frac{x+1}{5}$.

L[1].

1. Add $x+y-z$, $x-y+z$, $y+z-x$, and $z+x-y$.
2. From $4y-6z-x+10$ take $5y-6z+x+11$.
3. Multiply $x^4+x^2y^2+y^4$ by x^2-y^2.
4. Simplify $a-(4a+b)+(6a+b-10)$.
5. Divide x^6-y^6 by x^3-y^3.
6. Solve $\frac{3x-5}{2}-\frac{5x-9}{6}=\frac{4x-7}{3}$.

M[1].

1. Find value of $e^4+6e^2b^2+b^4-4e^3b-4eb^3$, when $e=5$, $b=2$.
2. Add $a^3-3a^2b+3ab^2-b^3$, $2a^3+5a^2b-6ab^2-7b^2$, and $a^3-ab^2+2b^3$.
3. Simplify $2a-(3b+2c)-[5b-(6c-6b)+5c-\{2a-(c+2b)\}]$.
4. Solve $\frac{x+4}{3}-\frac{x-4}{5}=2+\frac{3x-1}{15}$.
5. Multiply $a^2-ab+2b^2$ by $a^2+ab+2b^2$.
6. Divide $x^5+2x^4y+3x^3y^2-x^2y^3-2xy^4-3y^5$ by x^3-y^3.

N[1].

1. Find value of $\frac{2a+2}{a-3}+\frac{a+1}{a-2}+\frac{a^2-1}{a+3}$, when $a=5$.
2. From $7x^2-8x-1$ take $5x^2-6x+3$.
3. Simplify $16-\{5-2x-[1-(3-x)]\}$.

4. Solve $\frac{x-1}{2}-\frac{x-2}{3}+\frac{x-3}{4}=\frac{2}{3}$.
5. Multiply $x^5-x^4y+xy^4-y^5$ by $x+y$.
6. Divide $x^4+x^3-9x^2-16x-4$ by x^2+4x+4.

O¹.

1. Find value of $\frac{8a^2+3b^2}{a^2+b^2}+\frac{4c^2+6b^2}{c^2-b^2}-\frac{c^2+d^2}{e^2}$, when $a=1$, $b=2$, $c=3$, $d=4$, $e=5$.
2. From $4x^4-3x^3-2x^2-7x+9$ take $x^4-2x^3-2x^2+7x-9$.
3. Simplify $2a-\{2a-[2a-(2a-\overline{2a-a})]\}$.
4. Solve $\frac{x}{4}-\frac{5x+8}{6}=\frac{2x-9}{3}$.
5. Multiply $x^2+y^2-xy+x+y-1$ by $x+y-1$.
6. Divide x^4+64 by x^2+4x+8.

P¹.

1. Find value of $\frac{28}{a^2+b^2+c^2}+\frac{12}{d^2-c^2-b^2}+\frac{40}{a^2+e^2-d^2}$, when $a=1$, $b=2$, $c=3$, $d=4$, $e=5$.
2. Add $x^2+y^4+z^3$, $-4x^2-5z^3$, $8x^2-7y^4+10z^3$, and $6y^4-6z^3$.
3. Simplify $16-x-[7x-\{8x-(9x-\overline{3x-6x})\}]$.
4. Solve $\frac{3x+5}{7}-\frac{2x+7}{3}+10-\frac{3x}{5}=0$.
5. Multiply $x^4+2x^3y+4x^2y^2+8xy^3+16y^4$ by $x-2y$.
6. Divide $x^4+10x^3+35x^2+50x+24$ by x^2+5x+4.

Q¹.

1. Find value of $a^3+3a^2b+3ab^2+b^3$, when $a=1$ and $b=2$.
2. From $2x^2-2ax+3a^2$ take x^2-ax+a^2.
3. Simplify $2x-[3y-\{4x-(5y-\overline{6x-7y})\}]$.
4. Solve $\frac{1}{7}(3x-4)+\frac{1}{3}(5x+3)=43-5x$.
5. Multiply $81x^4+27x^3y+9x^2y^2+3xy^3+y^4$ by $3x-y$.
6. Divide $x^4+x^3-24x^2-35x+57$ by x^2+2x-3.

R¹.

1. Find value of $\frac{d^3}{b^5}$, when $b=2$, $d=4$.
2. Add $3x^2-4xy+y^2+2x+3y-7$, $2x^2-4y^2+3x-5y+8$, $10xy+8y^2+9y$ and $5x^2-6xy+3y^2+7x-7y+11$.
3. Simplify $2a-[3b+(2b-c)-4c+\{2a-(3b-\overline{c-2b})\}]$.
4. Solve $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}+\frac{x}{5}=7\frac{5}{6}$.
5. Multiply $x+2y-3z$ by $x-2y+3z$.
6. Divide $1-x-3x^2-x^5$ by $1+2x+x^2$.

S^1.

1. Find value of $\frac{e^3+b}{c^2-b^3}$, when $b=2$, $c=3$, $e=5$.
2. From $x^2-3xy-y^2+yz-2z^2$ take $x^2+2xy+5xz-3y^2-2z^2$.
3. Simplify $-(x-y)+4(x-8y)-(x+y)$.
4. Solve $\frac{x}{2}-\frac{x-2}{3}=\frac{x+3}{4}-\frac{2}{3}$.
5. Multiply $a^2-ax+bx+b^2$ by $a+b+x$.
6. Divide x^3+y^3 by $x+y$.

T^1.

1. Find value of $a(b+c)$, when $a=1$, $b=2$, $c=3$.
2. From $a^3-3a^2b+3ab^2-b^3$ take $-a^3+3a^2b-3ab^2+b^3$.
3. Simplify $4x+[8x+\{9x+(40x+\overline{39x+1})\}]$.
4. Solve $\frac{1}{3}(x-3)-\frac{1}{4}(x-8)+\frac{1}{6}(x-5)=0$.
5. Multiply a^2-ab+b^2 by $a+b$.
6. Divide $56x^2+113xy+56y^2$ by $8x+7y$.

TEST PAPERS.—SECOND YEAR.

A^2.

1. $\frac{5x-1}{2x+3}=\frac{5x-3}{2x-3}$. Find x.
2. Reduce $\frac{x^2-2x-3}{x^2-10x+21}$ to lowest terms by method of factors.
3. Solve $\frac{2x+3y}{5}=10-\frac{y}{3}$, $\frac{4y-3x}{6}=\frac{3x}{4}+1$.
4. $\frac{x^2+x-2}{x^2-7x}\div\frac{x^2+2x}{x^2-13x+42}$.
5. Simplify $\frac{1}{1-x}-\frac{1}{1+x}+\frac{2x}{1+x^2}$.
6. $\sqrt{x^4-2x^3+2x^2-x+\frac{1}{4}}$.

B^2.

1. Solve $\frac{2}{1-5x}-\frac{5}{1-2x}=0$.
2. Find G.C.M. of $2x^2-5x+2$, and $12x^3-8x^2-3x+2$.
3. Find L.C.M. of x^2-3x+2, x^2-5x+6, x^2-4x+3.
4. Solve $x-\frac{y-2}{7}=5$; $4y-\frac{x+10}{3}=3$.
5. Simplify $\frac{x}{1-x}-\frac{x^2}{1-x^2}+\frac{x}{1+x^2}$.
6. A father is 30 years old and his son is 6 years old. In how many years will the father's age be just twice the son's?

C².

1. Out of a cask of wine out of which $\frac{1}{6}$ had leaked away, 10 gallons were drawn, and then the cask was $\frac{2}{3}$ full. How much does it hold?
2. Solve $\frac{x}{3}-\frac{x^2-5x}{3x-7}=\frac{2}{3}$.
3. Solve $\frac{x+2}{3}+8y=31$, $\frac{y+5}{4}+10x=192$.
4. Write in factors x^3-y^3, x^3+y^3 and x^4-y^4.
5. Simplify $\frac{x+y}{y}-\frac{2x}{x+y}+\frac{x^2y-x^3}{y(x^2-y^2)}$.
6. $\sqrt{1-6x+13x^2-12x^3+4x^4}$.

D².

1. Solve $\frac{3}{x-5}=2-\frac{2x}{x-3}$.
2. Solve $\frac{9}{x}-\frac{4}{y}=2$, $\frac{18}{x}+\frac{8}{y}=10$.
3. Simplify $\frac{2a}{a+b}+\frac{2b}{a-b}+\frac{a^2+b^2}{b^2-a^2}$.
4. Reduce $\frac{(x^3-y^3)(x+y)}{(x^3+y^3)(x-y)}$ to lowest terms.
5. $\sqrt{49x^4-28x^3-17x^2+6x+\frac{9}{4}}$.
6. Find two numbers differing by 8, such that 4 times the less may exceed twice the greater by 10.

E².

1. Three persons, A, B, C, have £76; B has £10 more than A, C has as much as A and B together. How much has each?
2. Reduce $\frac{x^3-4x^2+9x-10}{x^3+2x^2-3x+20}$ to lowest terms.
3. Find the L.C.M. of $x^2-9x+20$, $x^2-12x+35$, and $x^2-11x+28$.
4. Solve $\frac{3x+2}{x-1}=5-\frac{2x-4}{x+2}$.
5. Solve $\frac{3x-5y}{2}+3=\frac{2x+y}{5}$; $8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3}$.
6. Simplify $\frac{a+b}{a-b}-\frac{a-b}{a+b}-\frac{a^2+b^2}{b^2-a^2}$.

F².

1. Solve $\frac{(2x+3)x}{2x+1}+\frac{1}{3x}=x+1$.
2. Solve $\frac{2x-y}{7}+3x=2y-6$, $\frac{y+3}{5}+\frac{y-x}{6}=2x-8$.

3. Find L.C.M. of $x^3 - x$, $x^3 - 1$, and $x^3 + 1$.
4. $\sqrt{x^4 + 8x^2 + 24 + \frac{16}{x^4} + \frac{32}{x^2}}$.
5. Write down in factors $9x^2 - 16y^4$.
6. A and B began to play with equal sums; A won £5, and then 3 times A's money was equal to 11 times B's money. What had each at first?

G².

1. Divide 50 into two such parts that the double of one part may be three times as great as the other.
2. Solve $\frac{3}{x+1} - \frac{x+1}{x-1} = \frac{x^2}{1-x^2}$.
3. Reduce $\frac{a^2 - a - 20}{a^2 + a - 12}$ to lowest terms.
4. Find L.C.M. of $a^2b - ab^2$, $a^2 + ab$, and $a^3 + b^3$.
5. Solve $\frac{x-2}{5} - \frac{10-x}{3} = \frac{y-10}{4}$, $\frac{2y+4}{3} = \frac{4x+y+13}{8}$.
6. Simplify $\frac{x^2 - y^2}{x^2 - 3xy + 2y^2} \times \frac{xy - 2y^2}{x^2 + xy} \times \frac{x^2 - xy}{(x-y)^2}$.

H².

1. Reduce $\frac{x^2 + 5x + 6}{x^3 + x + 10}$ to lowest terms.
2. $\sqrt{1 - 2x + 5x^2 - 4x^3 + 4x^4}$.
3. Solve $\frac{x+4}{3x-8} = \frac{x+5}{3x-7}$.
4. Simplify $\frac{1}{2x+1} + \frac{1}{2x-1} + \frac{4x}{1-4x^2}$.
5. Find L.C.M. of $x^2 - x - 6$, $x^2 + x - 2$, and $x^2 - 4x + 3$.
6. A number divided by the sum of its digits gives quotient 4. If you add 27 to the number the digits are reversed. Find the number.

I².

1. Solve $12x + z - 4y = 3$, $x - y + 1 = 2z$, $5x = 2y$.
2. $\sqrt{9x^4 + 12x^3 - 2x^2 - 4x + 1}$.
3. Simplify $\frac{x^2 + x - 6}{x^2 + 2x - 8} \times \frac{x^2 + 4x}{x^2 - 9}$.
4. Find the G.C.M. of $2x^4 - x^3 - 10x^2 - 11x + 8$, and $2x^3 - 3x^2 - 9x + 5$.
5. Simplify $\frac{a}{2(x-a)} - \frac{a}{2(x+a)} - \frac{a^4}{x^2(x^2 - a^2)}$.
6. At present A's age is $\frac{3}{4}$ of B's age. Eight years ago it was $\frac{2}{3}$ of B's. Find their ages.

J^2.

1. Solve $\frac{3x-1}{2x-1}-\frac{4x-2}{3x-1}=\frac{1}{6}$.
2. Solve $3x-\frac{y-5}{7}=\frac{4x-3}{2}$, $\frac{3y+4}{5}-\frac{1}{3}(2x-5)=y$.
3. Reduce $\frac{x^3-2x+1}{3x^3+7x-10}$ to its lowest terms.
4. Divide $\frac{ax-x^2}{(a+x)^2}$ by $\frac{x^2}{a^2-x^2}$.
5. $\sqrt{1-6x+13x^2-12x^3+4x^4}$.
6. If 6 lbs. of tea and 11 lbs. of sugar cost £1, 3*s*. 8*d*., and 11 lbs. of tea and 6 lbs. of sugar cost £1, 18*s*. 8*d*., find price of 1 lb. of tea and of 1 lb. of sugar.

K^2.

1. Solve $\frac{3x+2}{x-1}=5-\frac{2x-4}{x+2}$.
2. Solve $x-\frac{y-2}{7}=5$, $4y-\frac{x+10}{3}=3$.
3. Simplify $\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1}$.
4. Find the L.C.M. of x^2-x-6, x^2+x-2, and x^2-4x+3.
5. $\sqrt{x^4-2x^3+2x^2-x+\frac{1}{4}}$.
6. Divide £864 between A, B, and C, so that A gets $\frac{5}{11}$ of B's share, and C's share is equal to A's and B's together.

L^2.

1. Solve $\frac{6x+8}{2x+1}-\frac{2x+38}{x+12}=1$.
2. Solve $\frac{x+2}{3}+4y=2$, $\frac{y+5}{11}-\frac{x+1}{2}=1$.
3. Find the G.C.M. of $x^3-4x^2+9x-10$, and $x^3+2x^2-3x+20$.
4. Simplify $\frac{3}{1+y}+\frac{5}{1-y}-\frac{6}{1+y^2}$.
5. Resolve into factors $x^4-y\frac{1}{3}^4$, x^2-5x-6, $9x^2-24xy+16y^2$.
6. A purse of sovereigns is divided among three persons. The first receives $\frac{1}{2}$ of them and 1 more; the second $\frac{1}{2}$ the remainder and 1 more; the third gets 6: find the number the purse contained.

ANSWERS

I.

1. 19. **2.** 6. **3.** 28. **4.** 19. **5.** 64. **6.** 0. **7.** -14. **8.** 106. **9.** 120. **10.** 32. **11.** 4. **12.** 5. **13.** 7. **14.** 11. **15.** $\frac{17}{23}$. **16.** 1. **17.** 9. **18.** 11. **19.** 24. **20.** 10. **21.** 64. **22.** 73. **23.** 8. **24.** $\frac{2}{3}$. **25.** 0. **26.** 1. **27.** $\frac{125}{343}$. **28.** 28. **29.** 25. **30.** 2. **31.** 6. **32.** 6. **33.** 3. **34.** 15. **35.** 81. **36.** 4. **37.** 2,256. **38.** $\frac{1}{2}$.

II.

1. $8a-b$. **2.** $3ab+bc$. **3.** $2x+2y+2z$. **4.** $8a^2+2b^2$. **5.** $3x+11y+5z$. **6.** $8x-z-4$. **7.** $a^2+2ab+b^2$. **8.** $7x^3-13ax^2+15a^2x-2a^3$. **9.** a^2b+3ac^2. **10.** m^3-n^3. **11.** $2x^2+2y^2-4xy$. **12.** $4x^2+2y^4+xz^2+4yz^2$. **13.** 0. **14.** $5a^5+3a^3b^2-2a^2b^3+5b^5$. **15.** $7a^2+7ab+b^2$. **16.** $7x^2y$. **17.** $7a^3-5a^2b+2ab^2-2b^3$. **18.** $7x^3-9x^2+5x-4$. **19.** $-a+11b+4c$. **20.** $7q^2+9-2p^2-3pq$. **21.** $7a+2b-2c+2$. **22.** $3a-2b$. **23.** x^3+6x^2-x-11. **24.** $4x^3+3xy^2$. **25.** $6x+24y-6z$. **26.** $3a^2-2b^2+1$. **27.** $2a^2+2b^2-2c^2+4bc$. **28.** $13-9a$. **29.** $2ab^2c^3-5a^2b^3c+3a^3bc^2$. **30.** $4x^4+3x^3-7x^2-8x$.

III.

1. $2+6a$. **2.** $2b$. **3.** $-2b$. **4.** $4a-4b+4c$. **5.** $4x+y+5z$. **6.** $6ax-6ky+6cz$. **7.** $3m-n+p$. **8.** $4xy$. **9.** $-2b^2-4ab$. **10.** $-6x^2-2$. **11.** $4x^4-4x^2y^2-5y^4$. **12.** $-5x^3y+5x^2y^2-5xy^3$. **13.** $a+2b-c$. **14.** $a-b-c-d$. **15.** $a+b-5c+d$. **16.** $2xy+2yz-2xz+z^2$. **17.** $16x^4+11a^2x^2+3ax^3-11a^4$; $6x^4-2a^2x^2+6ax^3-27a^4$. **18.** $15y^4-3xy^2+4x^2-x^4$; $2x^4-2y^4-2x^2+2xy^2$. **19.** $2p^2-5pq+3q^2$. **20.** $6a-2$. **21.** $2x-2z$. **22.** $-6a^2-2$. **23.** $-6a+7b+2c-2d$. **24.** $-2ax$. **25.** $-x^4+x^3+x$. **26.** $-2a+2b+c-14d-e$. **27.** $2a^2x^2+6ax-7$. **28.** $5x^3+7x^2y-2xy^2-3y^3$. **29.** y. **30.** $2b-c$.

IV.

1. $20x^2y^6z^4$. **2.** $36ap^2q^3r^4x$. **3.** $30x^4y^4$. **4.** $-27a^3b^6c^4d$. **5.** $12a^3b^2cx^6y^7$. **6.** $a^4-a^3x+a^2x^2$. **7.** $54a^2x^2-42x^2-18b^2x^2$. **8.** $-a^3b+2a^2b^2-ab^3$. **9.** $6p^3q^2-6p^2q^3+2pq^4$. **10.** $18m^3n-$

$15m^2n - 18mn$. **11.** $35a^4b^2c^3 - 10a^3b^3c^3 + 5a^2b^2c^4$. **12.** $20x^7y^2 - 8x^5y^5 - 14x^2y^7$. **13.** $-acx^3 + bcx^2 - 3cx$. **14.** $4ap^2xy^2 - 8apqxy + 4aq^2x$. **15.** $3mn + 9m^2nx - 3mn^3x^2$. **16.** $3a^4b^2 + 11a^3b^3 - 4a^2b^4$. **17.** $6a^6b^4c^2 - 36a^5b^3c + 48a^4b^2$. **18.** $40x^6y^2 + 14x^5y^3 - 6x^4y^4$. **19.** $4x^3 + 12x^2 - 7x - 21$. **20.** $3a^3 - 3a^2b + 5ab^2 - 5b^3$. **21.** $a^3 + b^3$. **22.** $a^3 - b^3$. **23.** $x^3 - x^2 - x + 1$. **24.** $x^3 - 5x^2 - 4x + 20$. **25.** $8x^3 - 2x^2 + 13x + 35$. **26.** $6x^3 - x^2 - 22x + 15$. **27.** $1 + 9x + 26x^2 + 24x^3$. **28.** $1 - 12x + 47x^2 - 60x^3$. **29.** $x^3 + 3x^2y + 3xy^2 + {}^3y$. **30.** $x^3 - 3x^2y + 3xy^2 - y^3$. **31.** $10x^3 - 13x^2y + 10xy^2 - 3y^3$. **32.** $3a^3 - 17a^2bc + 22ab^2c^2 - 8b^3c^3$. **33.** $3m^6 + 8m^4n^2 - 15m^2n^4 + 4n^6$. **34.** $a^2 - b^2 + ac - bc$. **35.** $9x^2 - 6xy - 2yz - z^2$. **36.** $6p^3 + 2p^2q - 12pq + 15pr - 4q^2 + 5qr$. **37.** $-a^3x^3 + 2a^2x^2y - 4axy^2 + 3y^3$. **38.** $x^4 - 16$. **39.** $3x^4 - 20x^3 + 22x^2 + 11x - 10$. **40.** $8x^4 + 22x^2 - 6$. **41.** $a^4 + 2a^3b - 2ab^3 - b^4$. **42.** $b^4 - 2b^3 + 2b - 1$. **43.** $2x^4 - 7x^3y + 10x^2y^2 - 8xy^3 + 3y^4$. **44.** $a^3 + a^2b - ab^2 - b^3 - ac^2 + bc^2$. **45.** $a^2b + a^2c - b^3 + b^2c + bc^2 - c^3$. **46.** $6x^4 - 96$. **47.** $6a^8 - 96x^4$. **48.** $a^6 + a^5 + a + 1$. **49.** $x^6 - x^4y^2 + x^3y^3 - x^2y^4 + 2xy^5 - y^6$. **50.** $a^3 - 3abc + b^3 + c^3$. **51.** $a^2b + a^2c - ab^2 + abc + ac^2 - b^2c + bc^2$. **52.** $x^6 - 2a^3x^3 + a^6$. **53.** $x^3 + 30xyz - 8y^3 + 125z^3$. **54.** $1 - 6x^5 + 5x^6$. **55.** $a^3 - 18abc + 8b^3 + 27c^3$. **56.** $a^7 - 7a^6 + 21a^5 - 35a^4 + 35a^3 - 21a^2 + 7a - 1$. **57.** $1 + x^2 + x^4 - x^6 - x^8 - x^{10}$. **58.** $a^8 - 1$. **59** $a^8b^8 - x^8$. **60.** $x^6 - y^6$. **61.** $a^8 + a^4b^4 + b^8$. **62.** $1 - xy - xz - yz + x^2yz + xy^2z + xyz^2 - x^2y^2z^2$.

V.

1. $-3xy + xz + 4yz$. **2.** $2a^3 - 5a^2b + 3ab^2 + 4b^3$. **3.** $a - 3c + 2bcd$. **4.** $4ay - 3xy + 5b^2 - a^2cxy$. **5.** $4a^4c^3 - 3a^3bc^2d + 3 - 2a^2b$. **6.** $-b^3x^3y^3 + 2ab^2x^2y^2 + 3a^2bxy - 4a^3$. **7.** $1 - 3xy + 2x^2y^2z - 4ax^3y^3z^2$. **8.** $-a^2b^3 + b^3cd - 2ad^4 + 2bc$. **9.** $a^6 - 5a^4 - 9a^2 + 2$. **10.** $2ax^2 - 3a^2x^4 + 4a^3x^6 - 5a^4x^8$. **11.** $x + 7$. **12.** $x + 12$. **13.** $x - 9$. **14.** $x - 16$. **15.** $a - 6$. **16.** $a + 24$. **17.** $a - 4$. **18.** $a + 7$. **19.** $y - 8$. **20.** $2y + 11$. **21.** $3x - 2$. **22.** $5a + 2$. **23.** $4x + 2$. **24.** $7x + 3$. **25.** $5x - 1$. **26.** $3a - 5$. **27.** $12bc + 2$. **28.** $y^2 + 4$. **29.** $x^2 + 12$. **30.** $3x^2 - 9$. **31.** $4x^2 - 3xy + 2y^2$. **32.** $b + 5$. **33.** $x + 7y$. **34.** $a + 6b$. **35.** $4p + 3$. **36.** $x - 3$. **37.** $x - 4y$. **38.** $2ab + 4$. **39.** $x - 4$. **40.** $x - 6y$. **41.** $3x - 4y + 5z$. **42.** $3a + 2b + 3c$. **43.** $a + b - c$. **44.** $m + n - x$. **45.** $ax - by + 1$. **46.** $2x - 3y - 4z$. **47.** $3a - 2b + 4c$. **48.** $a + b - x + y$. **49.** $a^2 - ab + 2a + b^2 - 4b + 4$. **50.** $x - y + z$.

VI.

1. $7a-6$. **2.** $3-4a$. **3.** $5x^2-3x+7$. **4.** $8x-6$. **5.** $6x$. **6.** $16b$. **7.** $5a+5b+5c$. **8.** $3a-8b-4c$. **9.** $x+7y+8z$. **10.** $2a^2+2b^2$. **11.** $4ab$. **12.** $3x-1$. **13.** $a-2b+c$. **14.** $2a^2+3a+1$. **15.** $a-b+c-d$. **16.** $a^3-3a^2b—3ab^2+b^3$. **17.** $6x^2-5$. **18.** $6x-y$. **19.** $3x^3-3x^2+3x-6$. **20.** $2a+b-2c$. **21.** $6-5a$. **22.** $13-5x$. **23.** $5a$. **24.** $2x^3-x^2-2$. **25.** $9a+10b-13c$. **26.** $5-a$. **27.** $3a-15b-8c$. **28.** x^2-y^2. **29.** $1-x^2-x^4$. **30.** 50. **31.** 3. **32.** $7a-3b-3$. **33.** $1-x$. **34.** $24ab^5+36a^2b^4+72a^3b^3$. **35.** 0.

VII.

1. 14. **2.** 15. **3.** 4. **4.** 5. **5.** 3. **6.** 2. **7.** 4. **8.** 2. **9.** -7. **10.** $\dfrac{d-c}{a-b}$. **11.** 6. **12.** 33. **13.** 6. **14.** 7. **15.** $-1\frac{1}{4}$. **16.** 6. **17.** 24. **18.** 20. **19.** 6. **20.** 4. **21.** 36. **22.** 11. **23.** 2. **24.** 3. **25.** 5. **26.** $14\frac{4}{7}$. **27.** 7. **28.** 6. **29.** $-2\frac{3}{4}$. **30.** $1\frac{7}{25}$ **31.** $-2\frac{1}{2}$. **32.** $49\frac{8}{55}$. **33.** $-2\frac{21}{44}$. **34.** $-2\frac{54}{67}$. **35.** $-12\frac{9}{11}$. **36.** $1\frac{6}{35}$. **37.** -3. **38.** 7. **39.** 6. **40.** $35\frac{5}{7}$. **41.** 24. **42.** 2. **43.** 9. **44.** $\frac{1}{2}$. **45.** 4. **46.** 9. **47.** 35. **48.** 21. **49.** 7. **50.** $4\frac{23}{48}$. **51.** 4. **52.** $\frac{2}{3}$.

VIII.

1. 7. **2.** 14, 11. **3.** 20, 16. **4.** 4, 20. **5.** 20, 320. **6.** 40, 10. **7.** 9, 27, 81. **8.** 6. **9.** 24. **10.** 48. **11.** 30 lbs. **12.** £750. **13.** 144, 36. **14.** £2,000. **15.** 20. **16.** 126. **17.** 120, 60. **18.** No leakage. **19.** Officer £800, man £400, boy £200. **20.** Sheep £10, cow £30, horse £50. **21.** 36. **22.** 1,440. **23.** 4. **24.** 15, 20, 7.

IX.

1. $x^2+10x+21$. **2.** $x^2-11x+30$. **3.** $x^2+7x-18$. **4.** $x^2-5x-24$. **5.** $a^2+12a+36$. **6.** $x^3+6x^2+11x+6$. **7.** $x^3-6x^2+11x-6$. **8.** $x^3+3x^2-10x-24$. **9.** $x^3-4x^2-47x+210$. **10.** $x^3-79x-210$. **11.** 961, 1,764, 2,809, 3,721, 6,724, 10,609. **12.** 396, 891, 2,484, 4,899, 8,075, 6,399. **13.** $x^2+2xy+y^2$, $p^2+2pq+q^2$, x^2+2x+1, $4+4y+y^2$. **14.** $x^2+6xy+9y^2$, $16x^2+8xy+y^2$, $9x^2+24xy+16y^2$, $25x^2+80xy+64y^2$. **15.** $9a^2-6a+1$, $16b^2-56b+49$, $25-60c+36c^2$, $1-14d+49d^2$. **16.** $x^4+2x^3+3x^2+2x+1$, $x^4+4x^3+6x^2+4x+1$, $x^4+4x^3+8x^2+8x+4$. **17.** $4x^4+12x^3+25x^2+24x$

$+16$, $9x^4+30x^3+61x^2+60x+36$, $16x^4+56x^3+121x^2+126x+81$. **18.** $x^4-4x^3y+2x^2y^2+4xy^3+y^4$, $x^4-6x^3y+17x^2y^2-24xy^3+16y^4$, $x^4+8x^3y+6x^2y^2-40xy^3+25y^4$. **19.** $x^2+2xy+2xz+y^2+2yz+z^2$, $x^2-2xy+2xz+y^2-2yz+z^2$, $x^2-2xy-2xz+y^2+2yz+z^2$. **20.** $x^6+2x^5+3x^4+4x^3+3x^2+2x+1$, $x^6+4x^5+6x^4+8x^3+9x^2+4x+4$. **21.** $4x^6+12x^5+25x^4+28x^3+22x^2+8x+1$, $9x^6+30x^5+31x^4+34x^3+41x^2+8x+16$. **22.** $x^6-6x^5y+x^4y^2+28x^3y^3+4x^2y^4-16xy^5+4y^6$, $x^6-4x^5y-4x^4y^2+18x^3y^3+12x^2y^4-8xy^5+y^6$. **23.** $x^2+2xy+y^2-z^2$, $x^2-y^2+2xz+z^2$. **24.** $x^4+2x^3+x^2-1$, x^4+x^2+1. **25.** $a^4+2a^3b+a^2b^2-b^4$, $a^4+a^2b^2+b^4$. **26.** $a^2+2ab+b^2-c^2-2cd-d^2$. **27.** $a^2-2ab+b^2-c^2-2cd-d^2$. **28.** $x^6+2x^5+x^4-x^2-2x-1$. **29.** $x^6-4x^5+4x^4-9x^2-24x-16$. **30.** $2p^2q^2+2p^2r^2+2q^2r^2-p^4-q^4-r^4$.

X.

1. $(x+2)(x+3)$. **2.** $(x-4)(x-4)$. **3.** $(x+1)(x+5)$. **4.** $(a-5b)(a-7b)$. **5.** $(x+4)(x+4)$. **6.** $(a-7b)^2$. **7.** $(y^2-5)(y^2-8)$. **8.** $(a^2+2b)(a^2+7b)$. **9.** $(3+x)(7+x)$. **10.** $(6-x^2y^2)^2$. **11.** $(4x+y)(7x+y)$. **12.** $(x+y+5)(x+y+6)$. **13.** $(x+7)(x-3)$. **14.** $(x-6)(x+3)$. **15.** $(x+10)(x-1)$. **16.** $(x+9)(x-8)$. **17.** $(a+13b)(a-2b)$. **18.** $(a-9b)(a+5b)$. **19.** $(y-13z)(y+12z)$. **20.** $(xy+7)(xy-6)$. **21.** $(x^2+16y)(x^2-3y)$. **22.** $(y^3-8z^2)(y^3+5z^2)$. **23.** $(a+b-10)(a+b+6)$. **24.** $\{x+5(y+z)\}\{x-3(y+z)\}$. **25.** $\{x+y+5(x-y)\}\{x+y-(x-y)\}$. **26.** $(12xy+z)(12xy-z)$. **27.** $(9x^2y+11z^3)(9x^2y-11z^3)$. **28.** $(x^5+10y^{10})(x^5-10y^{10})$. **29.** $(1+7x^3y^2z)(1-7x^3y^2z)$. **30.** $(x+1)(x-1)$. **31.** $(a-b+c)(a-b-c)$. **32.** $(2a+3b+2c)(2a+3b-2c)$. **33.** $(x+y+z)(x-y-z)$. **34.** $(x+3y-4z)(x-3y+4z)$. **35.** $(c^2+a^2+ab-b^2)(c^2-a^2-ab+b^2)$. **36.** $(x+y+a-b)(x+y-a+b)$. **37.** $(a-b+c+d)(a-b-c-d)$. **38.** $(a+b+c-d)(a+b-c+d)$. **39.** $(1-xy+ab+cd)(1-xy-ab-cd)$. **40.** $(a-5b+c-6d)(a-5b-c+6d)$. **41.** $2a\times 2b$. **42.** $8a\times 6b$. **43.** 250×4. **44.** 600×2. **45.** 1000×134. **46.** $(xy+4)(x^2y^2-4xy+16)$. **47.** $(a+b+5c)(a^2+2ab+b^2-5ac-5bc+25c^2)$. **48.** $(2a+b-c)(4a^2-2ab+2ac+b^2-2bc+c^2)$. **49.** $2x(x^2+3y^2)$. **50.** $(6+4a-5c)(36-24a+30c+16a^2-40ac+25c^2)$. **51.** $(x+y-z)(x^2+2xy+y^2+xz+yz+z^2)$. **52.** $2b(3a^2+b^2)$. **53.** $6b(12a^2+9b^2)$. **54.** $(4-a+x)(16+4a-4x+a^2-2ax+x^2)$. **55.** $(x+2y)(x-2y)(x^2+2xy+4y^2)(x^2-2xy+4y^2)$. **56.** $(1+a)(1-a)(1+a+a^2)(1-a+a^2)$.

XI.

1. $2ax^2y^2$. **2.** $3a^3bc^2$. **3.** $15a^2b^2c^4x^2$. **4.** $9a^6c^3x^2y^3$. **5.** $5xy^2z$. **6.** $11a^2b^2c$. **7.** $3(b+c)$. **8.** $4(x+y)$. **9.** $17(x+a)$. **10.** $xy(x+2)$. **11.** $x-a$. **12.** $x+3$. **13.** $x-4$. **14.** $x+8$. **15.** $x+3$. **16.** $2x+3$. **17.** $ax+1$. **18.** $x+3a$. **19.** $m-2n$. **20.** $2x-5$. **21.** $2x+3$. **22.** $5x-7$. **23.** $3x^2+1$. **24.** x^2+x+1. **25.** x^2-2x+2. **26.** x^2-6x-7. **27.** $6x^2+2x+13$. **28.** $2x^2+2x+3$. **29.** x^2-3x+4. **30.** $2x+7$. **31.** $9xy+6y^2$. **32.** $3x+2$. **33.** x^2-x-1. **34.** x^2+5x+6. **35.** x^2-1. **36.** $2x+3$.

XII.

1. $10a^3b^3c^3$. **2.** $36a^2b^2c^2$. **3.** $20a^2b^2x^3$. **4.** $126a^2b^4xy^6$. **5.** $60a^2bc^2$. **6.** $84a^3b^4c^2x^3y^2z^2$. **7.** $60a^2b^2cx^4y^4z^2$. **8.** $(x-2)(x^2-9)$. **9.** x^3-y^3. **10.** $(x+y)(x^3+y^3)$. **11.** $ab(a+b)$. **12.** $12ab(a+b)(a^3-b^3)$. **13.** $9-x^2$. **14.** $2xy(x^4-y^4)$. **15.** $(a+b)(b+c)(c+a)$. **16.** $(x-3)^2(x+3)^2$. **17.** $(x-1)(x-2)(x-5)(x-6)$. **18.** $6(x-2)^2(x^2+2x+4)$. **19.** $x(1-x)(1+x)^2$. **20.** $(x-1)(x+2)(x+3)$. **21.** $(x-2)(x^2-2r+2)(x^2-x+1)$. **22.** $(x-y)(x^2-2xy+2y^2)(x^2-2y^2)$. **23.** $x(x-1)(x+5)(x^2+x-5)$. **24.** $(x-1)(x^2+5x+5)(x^2+x-2)$. **25.** $(2x-3)^2(4x^2+6x+9)$. **26.** $(x+3y)(x^2+4)(x^2+x+1)$. **27.** $(x-2y)(x^2+3xy-5y^2)(x^2-2xy-2y^2)$. **28.** $(x+4)(x^2-3x+2)(x^2+5x-3)$. **29.** $(x-3)(x^2-x-1)(x^2+2x-1)$. **30.** $4(x+3)(x-5)(x^2-5x-3)$. **31.** $3x(x+1)(x-4)(x^3-2x^2-3x+4)$. **32.** $(x^2+xy-y^2)(x+2y)(x^2-xy-2y^2)$. **33.** $(x^2-5x+4)(x^2+3x-2)(x^2-x+3)$. **34.** $(x-1)(x^3+x^2+x+1)(x^3+2x^2+3x+4)$. **35.** $(x^2-2xy+5y^2)(x^2+2y^2)(x^2-3y^2)$. **36.** $(x^2+3x-4)(x+4)(x-3)(3x+4)$. **37.** $(x^2+1)(x^2-1)(x+2)(x-3)$. **38.** $(x^2-3x+2)(x-1)(x-2)(x-3)$. **39.** $2(x^2-y^2)(x^2-3y^2)(x-2y)$. **40.** $12(x+y)(x^2-xy+y^2)(x^3-x^2y+xy^2-y^3)(x^4-x^3y+x^2y^2-xy^3+y^4)$.

XIII.

1. $\frac{2a^3c^2x^3y^2}{3b^3d^2}$. **2.** $\frac{2a-b}{3ab}$. **3.** $\frac{3xz}{x^2-2yz}$. **4.** $\frac{4x^2}{3y}$. **5.** $\frac{2a^3b-a^2b^3c}{3ac^2+4b^2c^3}$. **6.** $\frac{4a}{3b}$. **7.** $\frac{a-b}{4a}$. **8.** $\frac{3x}{x-2y}$. **9.** $\frac{x-1}{3x}$. **10.** $\frac{5}{x-4}$. **11.** $\frac{a^2+2ab+4b^2}{3a^2}$. **12.** $\frac{1}{a-b}$. **13.** $\frac{1}{a^2+ab+b^2}$.

14. $\frac{x^2-2xy+4y^2}{x+2y}$. 15. $\frac{x+4}{x+7}$. 16. $\frac{1}{x-5}$. 17. $\frac{a+3b}{a^2+3ab+9b^2}$.

18. $\frac{a+12b}{a+9b}$. 19. $\frac{a-1}{a+5}$. 20. $\frac{a+b+c}{a-b+c}$. 21. $\frac{1}{a-b-c}$.

22. $\frac{1-x-y}{1+x-y}$. 23. $\frac{x+a}{x-a}$. 24. $\frac{a^2-ab+b^2}{a+c}$. 25. $\frac{x^3-2x^2-3}{x^2+x-1}$.

26. $\frac{x^2+3x-2}{x^2-x+3}$. 27. $\frac{a^2-5ab+6b^2}{a+b}$. 28. $\frac{6x^3+6ax^2-a^2x-a^3}{4x^3+4ax^2-5a^2x+a^3}$.

29. $\frac{2a+5b}{5a^2+3b^2}$. 30. $\frac{b^2(3a+1)}{4a^2+2a-1}$.

XIV.

1. $\frac{6a}{12}, \frac{4b}{12}, \frac{3c}{12}$. 2. $\frac{30x}{60}, \frac{20x}{60}, \frac{15x}{60}, \frac{12x}{60}$. 3. $\frac{6acd}{12bcd}, \frac{3b^2d}{12bcd}, \frac{2bc^2}{12bcd}$.

4. $\frac{yz^2}{x^3y^3z^3}, \frac{2x^2z}{x^3y^3z^3}, \frac{3xy^2}{x^3y^3z^3}$. 5. $\frac{30ay^2z^2}{45x^2y^2z^2}, \frac{27bx^2z}{45x^2y^2z^2}, \frac{20cxy^2}{45x^2y^2z^2}$. 6. $\frac{a-b}{a^2-b^2}, \frac{a+b}{a^2-b^2}, \frac{a}{a^2-b^2}$.

7. $\frac{a^2+x^2}{a^3-x^3}, \frac{a^2+ax+x^2}{a^3-x^3}, \frac{a^2-ax}{a^3-x^3}$. 8. $\frac{3x^2+6x}{(x+2)(x^2-4)}, \frac{4x^2-16}{(x+2)(x^2-4)}, \frac{5x^2+20x+20}{(x+2)(x^2-4)}, \frac{3x^2-6x}{(x+2)(x^2-4)}$.

9. $\frac{x^2+6x+9}{(x+2)(x+3)(x+4)}, \frac{x^2+3x-4}{(x+2)(x+3)(x+4)}, \frac{x^2+4x+4}{(x+2)(x+3)(x+4)}$.

10. $\frac{ab-ac}{(a-b)(a-c)(b-c)}, \frac{ab-bc}{(a-b)(a-c)(b-c)}, \frac{ac-bc}{(a-b)(a-c)(b-c)}$. 11. $\frac{9a^2-a}{a^2-1}$.

12. $\frac{8x+1}{(x+2)(x-3)}$. 13. $\frac{14x^2+3x}{4x^2-9}$. 14. $\frac{2x^2}{x-y}$. 15. $\frac{13a^2+5a}{4(a+5)}$.

16. $\frac{x^3+x^2+xy^2-y^2}{x(x^2-y^2)}$. 17. $\frac{7x^2-5y^2}{(x+y)(x^2-y^2)}$. 18. $\frac{6x^2+3x+9}{x^3+27}$.

19. $\frac{9x^2+7ax+10a^2}{6(x^2-a^2)}$. 20. $\frac{6x-10}{(x-7)(x^2-4)}$. 21. $\frac{3x^3-x}{x^2-1}$.

22. $\frac{3a^4+3a^3+a^2-a-2}{a^3-1}$. 23. $\frac{3x^3+6x^2-11x-18}{(x+1)(x+2)(x+3)}$. 24. $\frac{2x^3+x^2y+xy^2+2y^3}{xy(x+y)}$.

25. $\frac{a+5b}{7}$. 26. $\frac{9x-26}{30}$. 27. $\frac{-3}{(x-2)(x-5)}$.

28. $\frac{2x-1}{x(x-1)}$. 29. $\frac{2xy+y^2}{x^2-y^2}$. 30. $\frac{-2b^3}{a^4+a^2b^2+b^4}$. 31. $\frac{-ax}{x^3+a^3}$.

32. $\frac{16-2x}{(x-1)(x-5)(x+6)}$. 33. $\frac{2x}{(x+1)(x+2)}$. 34. $\frac{-8x-23}{x^2+6x+5}$.

35. 0. 36. $\frac{3x^2-38}{(x-2)(x-3)(x-5)}$. 37. $\frac{5x+5}{x(x+2)(x+3)}$. 38. $\frac{x^2+1}{x^3-1}$.

39. $\frac{10x}{x^3-125}$. **40.** $\frac{-3x^2-27}{(x-3)(x^2+27)}$. **41.** $\frac{12a^2}{(a+5x)^3}$. **42.** $\frac{3a^4+6a^2x^2-x^4}{(a-x)^2(a+x)^2}$. **43.** $\frac{8ab(a^2+b^2)}{a^4+a^2b^2+b^4}$. **44.** $\frac{2a+c-3b}{(a+b)(b+c)(a+c)}$.
45. $\frac{4a}{a+b}$. **46.** $\frac{ax-x^2}{x^2-4a^2}$. **47.** $\frac{2x-a-b}{(x-a)(x-b)(a-b)}$. **48.** $\frac{x}{(x-a)(x-b)}$. **49.** $\frac{a^3+a^2b+ab^2+b^3+(a^2+ab+b^2)x}{(x+a)(x+b)}$. **50.** 0.

XV.

1. $\frac{3}{5}$. **2.** $\frac{(a-x)(x-y)}{a^2+ax+x^2}$. **3.** $\frac{3x+9}{2x^2-2x}$. **4.** $\frac{x^3-y^3}{x^3+y^3}$. **5.** 2.
6. $\frac{8z}{3xy+6y^2}$. **7.** $\frac{x+y}{x}$. **8.** $3x^2-9x$. **9.** $\frac{3(x+y)^3}{2(x-y)^2}$. **10.** $\frac{x(a-x)^2}{3}$.
11. 1. **12.** $\frac{x-4}{x+5}$. **13.** 1. **14.** $\frac{ax}{4(a+x)}$. **15.** $\frac{a+b+c}{a-b-c}$.
16. -1. **17.** $\frac{(x+y-z)^2}{(x-y-z)^2}$. **18.** $\frac{(x+a)^2}{(x-b)^2}$. **19.** $\frac{b^2x^2-c^2f^2}{x^2-1}$.
20. $\frac{x^2(x-4y)}{(x-5y)(x^2-5xy+25y^2)}$. **21.** $x+4$. **22.** $\frac{x(x+4y)}{3y(x+2y)}$. **23.** 1.
24. $\frac{p+q-r}{p-q+r}$. **25.** $\frac{2x+y+z}{2x-y-z}$.

XVI.

1. ab. **2.** $\frac{b}{a}$. **3.** $\frac{ab}{a+b}$. **4.** $\frac{a-b}{a+b}$. **5.** $a+1$. **6.** $\frac{cd}{c+d}$.
7. $a+b$. **8.** $\frac{a^2+6a+9}{a^2+2a+3}$. **9.** $4a$. **10.** $\frac{7}{8}$. **11.** 5. **12.** -9.
13. -2. **14.** -15. **15.** $(a-b)$. **16.** $\frac{ab-cd}{a+b+c+d}$. **17.** $\frac{1}{mn}$.
18. $-\frac{14}{5}$. **19.** 2. **20.** 2. **21.** -8. **22.** 5. **23.** 1.
24. 9. **25.** 15. **26.** 4. **27.** 16. **28.** 6. **29.** $-\frac{11}{8}$.
30. $-19\frac{1}{12}$.

XVII.

1. 27, 21. **2.** 16, 12. **3.** 7, 4. **4.** 12, 13. **5.** 56 boys; 88 girls. **6.** £20. **7.** 30. **8.** 15 years. **9.** 10 gallons.
10. 57, 13. **11.** 60. **12.** £12,000. **13.** 12 miles; 24 miles.
14. 52 years. **15.** £2. **16.** 7*s.* 4*d.*; 3*s.* 8*d.* **17.** 36 miles.
18. 5 gallons. **19.** 80 yards; 60 yards. **20.** £400.

XVIII.

1. 7, 4. **2.** 4, 5. **3.** 6, 5. **4.** 7, 4. **5.** 8, 9. **6.** 7, 6.
7. 1, 10. **8.** 3. 2. **9.** 15, 13. **10.** 121, 111. **11.** 102, 97.
12. $1\frac{1}{2}$, $1\frac{1}{3}$. **13.** $\frac{1}{4}$, $\frac{1}{7}$. **14.** $\frac{1}{2}$, $\frac{1}{3}$. **15.** $\frac{1}{6}$, $\frac{1}{9}$. **16.** $\frac{1}{3}$, $\frac{2}{5}$.
17. 0, 1. **18.** 4, $\frac{1}{2}$. **19.** $\frac{1}{11}$, $\frac{1}{12}$. **20.** 18, 24. **21.** 4, 9.
22. 11, 7. **23.** 16, 3. **24.** 8, 5. **25.** 15, 14. **26.** 27, 12.
27. -6, 14. **28.** 2, 3. **29.** 1, 1. **30.** 60, 40. **31.** 2, 1, 4.
32. 3, 2, 1. **33.** 1, 3, 5. **34.** 3, 1, 6. **35.** 2, 5, 8. **36.** 4, 5, 6.

XIX.

1. 87, 43. **2.** 15, 11. **3.** 18, 15. **4.** 20, 5. **5.** 12, 10.
6. $\frac{3}{8}$. **7.** 5*s.*, 3*s.* **8.** 2 miles per hour.

XX.

1. $x+y$. **2.** $2+b$. **3.** $3a+1$. **4.** $5b+12$. **5.** $3x+4y$.
6. $p-q$. **7.** $x-4y$. **8.** $8x-9y$. **9.** $7-6c$. **10.** x^2+x+1.
11. $x^2+3xy+4y^2$. **12.** $3x^2+5x+6$. **13.** x^2-2x+1. **14.** $x^2+4xy-5y^2$. **15.** $3-x+2x^2$. **16.** $x-y+z$. **17.** $4x-3y-2z$.
18. $3ax-2by+cz$. **19.** x^3+2x^2+x+2. **20.** $2x^3+3x^2y+4xy^2+y^3$.
21. $x^3-3x^2y-4xy^2+2y^3$. **22.** $3x^3-2x^2-x-4$. **23.** $x+\frac{a}{3}$.
24. $\frac{x}{2}-y$. **25.** $\frac{a}{3}-\frac{b}{4}$. **26.** $\frac{x^2}{4}+x+\frac{1}{3}$. **27.** $\frac{1}{2}-2x \quad \frac{3x^2}{4}$.
28. $x^2+\frac{x}{2}+\frac{1}{2}+\frac{1}{x}$.

XXI.

1. $(8x-1)(3x-2)$.
2. $(x-6)(8x-1)$.
3. $(9x-1)(2x+5)$.
4. $(3x+1)(2x-15)$.
5. $(x-10)(9x+1)$.
6. $(5x-1)(4x-3)$.
7. $(5x-1)(4x+3)$.
8. $(10x-3)(x-2)$.
9. $(2x-7)(7x-2)$.
10. $(7x+1)(2x-14)$.
11. $(7x-1)(7x+4)$.
12. $(x-1)(15x-8)$.
13. $(x-1)(15x+8)$.
14. $(5x-3)(x-8)$.
15. $(x-1)(13x-12)$.
16. $(x-2)(13x+6)$.

XXII.

1. $(x+y)(11x-24y)$.
2. $(2x+y)(11x+12y)$.
3. $(3x-2y)(11x+4y)$.
4. $(x+y)(9x+16y)$.
5. $(x+3y)(3x-16y)$.
6. $(6x-y)(3x+8y)$.
7. $(5x+2y)(2x+5y)$.
8. $(x+y)(4x-25y)$.
9. $(x-1)(25x-4y)$.
10. $(8x-3y)(x-4y)$.
11. $(4x+y)(8x-3y)$.
12. $(x-8y)(4x+3y)$.
13. $(x-y)(16x+15y)$.
14. $(x+2y)(8x+15y)$.
15. $(8x-y)(3x-10y)$.
16. $(3x+y)(2x-9y)$.
17. $(x-y)(2x-27y)$.
18. $(3x+y)(x+18y)$.

FIRST YEAR TEST PAPERS.

A¹.—**1.** 6. **2.** $a-c$. **3.** $2x^3-6x^2-2x-3$. **4.** $2a^3b^3-7a^2b^2c+13abc^2-5c^2$. **5.** $a^3-2a^2b+4ab^2-8b^3$. **6.** $x=8$.

B¹.—**1.** 162. **2.** $3a^2-4ab-b^2$. **3.** $a+3$. **4.** $x^4-x^3y-xy^3+y^4$. **5.** $2x^2-xy-3y^2$. **6.** $x=3$.

C¹.—**1.** 4. **2.** $6m^2+6n^2$. **3.** $2x-4$. **4.** x^3+2x^2-5x-6. **5.** x^2-px+q^2. **6.** $x=2$.

D¹.—**1.** 1. **2.** $3x^2-5xy+5y^2$. **3.** $7x^2+x+1$. **4.** $6x^4+5x^3y+6x^2y^2-17xy^3+6y^4$. **5.** $x^4+x^3y-xy^3-y^4$. **6.** $x=1$.

E¹.—**1.** 3. **2.** $3x+y$. **3.** $5a^3-3a^2-3a+2$. **4.** $7-9x$. **5.** $x^2-4y^2+12yz-9z^2$. **6.** $x=4$.

F¹.—**1.** -160. **2.** $-3y+2z$. **3.** $2a+2$. **4.** $16a^4-b^4$. **5.** x^2-y^2. **6.** $x=6$.

G¹.—**1.** See Definitions. **2.** $\frac{9}{35}$. **3.** $4-4a+3b+2c$. **4.** $x^4+x^2y^2+y^4$. **5.** $a^3+{}^2-2a$. **6.** $x=5$.

H¹.—**1.** 4. **2.** $-a+b+c$. **3.** $15-2a-5b$. **4.** a^6-2a^3+1. **5.** $x-y+z+1$. **6.** $x=-4$.

I¹.—**1.** $1\frac{7}{17}$. **2.** $1-a$. **3.** $32x^5-2x$. **4.** $x^2-xy+y^2-xz-yz+z^2$. **5.** $8x+2z$. **6.** $x=8$.

J¹.—**1.** 20. **2.** $(5-a)x+(5-2b)y$. **3.** $(a-1)x-(1-b)y+(c-3)z$. **4.** $1-6x^5+5x^6$. **5.** $x^4-x^3y+x^2y^2-xy^3+y^4$. **6.** $x=-2$.

K¹.—**1.** $ab-4bc-2cd+de$. **2.** 20. **3.** $(x^4-2x^3y+5x^2y^2-4xy^3+4y^4)$. **4.** x^2+xy+y^2. **5.** $2x$. **6.** $x=14$.

L¹.—**1.** $2x+2z$. **2.** $-2x-y-1$. **3.** x^6-y^6. **4.** $3a-10$. **5.** x^3+y^3. **6.** $x=2$.

M¹.—**1.** 81. **2.** $4a^3+2a^2b-4ab^2+b^3-7b^2$. **3.** $4a-16b-2c$. **4.** $x=3$. **5.** $a^4+3a^2b^2+4b^4$. **6.** $x^2+2xy+3y^2$.

N¹.—**1.** 11. **2.** $2x^2-2x-4$. **3.** $9+3x$. **4.** $x=3$. **5.** $x^6-x^4y^2+x^2y^4-y^6$. **6.** x^2-3x-1.

O¹.—**1.** 15. **2.** $3x^4-x^3-14x+18$. **3.** a. **4.** $x=1\frac{1}{3}$. **5.** $x^3+y^3+3xy-2x-2y+1$. **6.** x^2-4x+8.

P¹.—**1.** 10. **2.** $5x^2$. **3.** $16-12x$. **4.** $x=10$. **5.** x^5-32y^5. **6.** x^2+5x+6.

Q¹.—**1.** 27. **2.** $x^2-ax+2a^2$. **3.** $12x-15y$. **4.** $x=6$. **5.** $243x^5-y^5$. **6.** x^2-x-19.

R¹.—**1.** 2. **2.** $10x^2+8y^2+12x+12$. **3.** $4c$. **4.** $x=10$. **5.** $x^2-4y^2+12yz-9z^2$. **6.** $1-3x+2x^2-x^3$.

S¹.—**1.** 127. **2.** $-5xy-5xz+2y^2+yz$. **3.** $2x-32y$. **4.** $x=7$. **5.** $a^3+a^2b+ab^2+b^3+2b^2x-(a-b)x^2$. **6.** x^2-xy+y^2.

T¹.—**1.** 5. **2.** $2a^3-6a^2b+6ab^2-2b^3$. **3.** $100x+1$. **4.** $x=0$. **5.** a^3+b^3. **6.** $7x+8y$.

SECOND YEAR TEST PAPERS

A².—1. $x = \frac{6}{13}$. 2. $\frac{x+1}{x-7}$. 3. $x=4$, $y=9$. 4. $\frac{(x-1)(x-6)}{x^2}$.
5. $\frac{4x}{1-x^4}$. 6. $x^2 - x + \frac{1}{2}$.

B².—1. $x = \frac{1}{7}$. 2. $2x - 1$. 3. $(x-1), (x-2), (x-3)$. 4. $x=5, y=2$.
5. $\frac{2x}{1-x^4}$. 6. 18 years.

C².—1. 75 gal. 2. $x = -7$. 3. $x=19, y=3$. 4. $(x-y)(x^2+xy+y^2)$, $(x+y)(x^2-xy+y^4)$, $(x^2-y^2)(x^2+y^2)$. 5. $\frac{y}{x+y}$.
6. $1 - 3x + 2x^2$.

D².—1. $x = 4\frac{1}{3}$. 2. $x = \frac{18}{7}$, $y = \frac{8}{3}$. 3. $\frac{a^2+b^2}{a^2-b^2}$. 4. $\frac{x^2+xy+y^2}{x^2-xy+y^2}$.
5. $7x^2 - 2x - \frac{3}{2}$. 6. 21 and 13.

E².—1. £14, £24, £38. 2. $\frac{x-2}{x+4}$. 3. $(x-4), (x-5), (x-7)$.
4. $x=6$. 5. $x=12, y=6$. 6. $\frac{a^2+4ab+b^2}{a^2-b^2}$.

F².—1. $x=1$. 2. $x=6, y=12$. 3. $x(x^6-1)$. 4. $x^2+4+\frac{4}{x^2}$.
5. $(3x+4y^2), (3x-4y^2)$. 6. £8, 15*s*.

G².—1. 30, 20. 2. $x=4$. 3. $\frac{a-5}{a-3}$. 4. $ab(a-b)(a^3+b^3)$.
5. $x = \frac{3201}{708}$. 6. $\frac{y}{x-y}$.

H².—1. $\frac{x+3}{x^2-2x+5}$. 2. $1 - x + 2x^2$. 3. $x=6$. 4. 0.
5. $(x-1)(x-3)(x+2)$. 6. 36.

I².—1. $x=2, y=5, z=7$. 2. $3x^2+2x-1$. 3. $\frac{x}{x-3}$. 4. $2x-1$.
5. $\frac{a^2}{x^2}$. 6. A=14, B=35.

J².—1. $x = \frac{7}{17}$. 2. $x = -1\frac{1}{13}$, $y = 7\frac{25}{26}$. 3. $\frac{x-1}{3x^2+3x+10}$.
4. $\frac{(a-x)^2}{x(a+x)}$. 5. $1 - 3x + 2x^2$. 6. Sugar 4*d*. per lb., tea 4*s*. per lb.

K².—1. $x=6$. 2. $x=5, y=2$. 3. $\frac{-2}{x(4x^2-1)}$. 4. $(x-3)(x-1)(x+2)$. 5. $x^2 - x + \frac{1}{2}$. 6. A, £135; B, £297; C, £432.

L².—1. $x = 2$. 2. $x = -2$, $y = \frac{1}{2}$. 3. $x^2 - 2x + 5$.
4. $\frac{2+2y+14y^2+2y^3}{1-y^4}$. 5. $(x^2 - \frac{4}{3}y^2)(x^2 + \frac{4}{3}y^2)$, $(x-6)(x+1)$, $(3x-4y)(3x-4y)$. 6. 30.